The Infinite Actuary Exam 4/C Online Seminar
A.1. Probability Review Solutions
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1. From the tables, the mode is $\theta/2 = 10{,}000$, so $\theta = 20{,}000$.
To find the median, set $0.5 = F(x) = e^{-20{,}000/x}$ so $x = -20,000/\ln(0.5) = \boxed{28{,}854}$

Alternatively, the median is $\mathrm{VaR}_{0.5}(X) = \theta(-\ln 0.5)^{-1} = 28{,}854$

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2. $H(5) = \displaystyle\int_0^5 h(x)dx = \int_0^2 0\,dx + \int_2^5 \frac{z^2}{2x}dx = \frac{z^2}{2}\ln(5) - \frac{z^2}{2}\ln(2) = \frac{z^2}{2}\ln(2.5)$
$F(5) = 0.84 \Rightarrow S(5) = 1 - 0.84 = 0.16 = e^{-H(5)}$ and $H(5) = -\ln(0.16) = \dfrac{z^2}{2}\ln(2.5)$ so $\boxed{z = 2}$

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3. $f(y) = cy^2, 0 < y < 3$, so $1 = \displaystyle\int_0^3 cy^2 dy = c\left.\frac{y^3}{3}\right|_0^3 = 9c.$
That gives us $c = 1/9$, and for the 80th percentile, $0.8 = \displaystyle\int_0^t \frac{1}{9}y^2 dy = \frac{t^3}{27}$ so $\boxed{t = 2.78}$

4.

$$\begin{aligned}
S(t) &= e^{-H(t)} \\
-\ln S(t) = H(t) &= \int_0^{0.4} A + e^{2x}dx \\
-\ln(0.5) &= 0.4A + \frac{1}{2}\left(e^{0.8} - 1\right) \\
A &= \boxed{0.2}
\end{aligned}$$

5.

$$\begin{aligned}
H(x) &= \int_0^2 0\,dt + \int_2^x \frac{3}{t}dt = 3\ln\left(\frac{x}{2}\right) \qquad x > 2 \\
S(x) = e^{-H(x)} &= \begin{cases} 1 & x \le 2 \\ \dfrac{8}{x^3} & x > 2 \end{cases} \\
\mathrm{E}[X] &= \int_0^2 1\,dx + \int_2^\infty \frac{8}{x^3}dx = 2 + \frac{4}{2^2} = \boxed{3} \\
\text{or: } f(x) = -S'(x) &= \frac{24}{x^4} \qquad x > 2 \\
\mathrm{E}[X] &= \int_2^\infty x \cdot \frac{24}{x^4}dx = \int_2^\infty \frac{24}{x^3}dx = \frac{12}{2^2} = \boxed{3}
\end{aligned}$$

Or: $X \sim$ single parameter Pareto$(\alpha = 3, \theta = 2)$ and $\mathrm{E}[X] = \dfrac{\alpha\theta}{\alpha - 1} = \dfrac{3 \cdot 2}{2} = 3$

6.

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^\infty S(x)\,dx = \int_0^1 1\,dx + \int_1^2 (2 - x)dx \\
&= 1 + 2 - \frac{2^2 - 1^2}{2} = \boxed{1.5}
\end{aligned}$$

$$\text{or: } f(x) = -S'(x) = 1 \quad 1 < x < 2$$

$$\mathrm{E}[X] = \int_1^2 x \cdot 1\, dx = \frac{2^2 - 1^2}{2} = \boxed{1.5}$$

7.

$$f(x) = -S'(x) = \frac{2x}{3} \quad 1 < x < 2$$

$$\mathrm{E}[X] = \int_1^2 x \cdot \frac{2x}{3}\, dx = \frac{2}{9}\left(2^3 - 1^3\right) = \frac{14}{9}$$

$$\mathrm{E}[X^2] = \int_1^2 x^2 \cdot \frac{2x}{3}\, dx = \frac{2}{12}\left(2^4 - 1^4\right) = \frac{5}{2}$$

$$\mathrm{Var}[X] = \frac{5}{2} - \left(\frac{14}{9}\right)^2 = \frac{13}{162} = \boxed{0.080}$$

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8. $\sigma^2 = \dfrac{(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2}{5} = 2$ and $\mu_4 = \dfrac{(-2)^4 + (-1)^4 + 0^4 + 1^4 + 2^4}{5} = \dfrac{34}{5}$
so the kurtosis is $(34/5)/4 = \boxed{1.7}$

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9. From the exam tables, the mean of a Weibull distribution is $\theta\Gamma\left(1 + \dfrac{1}{\tau}\right)$. To evaluate that, we will use the fact that $\Gamma(x) = (x-1)!$ when x is an integer. Since only 60% of claims result in a payment, we have $\mathrm{E}[\text{cost}] = 0.6 \cdot \mathrm{E}[\text{Weibull}] = 0.6 \cdot 30(3-1)! = 36$ so (i) is false and we can eliminate A and D.

The survival function at 60 is 0.6 times the survival function of a Weibull at 60, giving us $0.6\left[e^{-(60/30)^{1/2}}\right] = 0.6 \cdot 0.243 = 0.146$ so (ii) is false and we can eliminate B and E.

We now know the answer is $\boxed{C}$ (only choice left!) but let's confirm that (iii) holds:
$h(60) = \dfrac{f(60)}{S(60)} = \dfrac{0.6\left[\frac{1}{2}\left(\frac{60}{30}\right)^{1/2}\frac{1}{60}e^{-\sqrt{2}}\right]}{0.6e^{-\sqrt{2}}} = 0.01179.$

10.

$$\mu = \frac{1}{10}\left[2 \cdot 400 + 7 \cdot 800 + 1600\right] = 800$$

$$\sigma^2 = (0.2)(-400)^2 + (0.1)(800)^2 = 96{,}000$$

$$\mu_3 = (0.2)(-400)^3 + (0.1)(800)^3 = 38{,}400{,}000$$

The skewness is thus $38{,}400{,}000/(96{,}000)^{3/2} = \boxed{1.29}$

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11. $\mathrm{Sk}(cX) = \mathrm{Sk}(X)$ for $c > 0$, so (i) is false.
$\mathrm{SD}[-Y] = \mathrm{SD}[Y]$ but $\mathrm{E}\left[\left((-Y) - \mathrm{E}[-Y]\right)^3\right] = -\mathrm{E}\left[(Y - \mathrm{E}[Y])^3\right]$ so $\mathrm{Sk}(-Y) = -\mathrm{Sk}(Y)$ and (ii) is true.
Since X and Y are iid with mean 0,

$$\mathrm{Sk}(X+Y) = \frac{\mathrm{E}\left[(X + Y - 0 - 0)^3\right]}{(\mathrm{SD}[X+Y])^3}$$

$$= \frac{\mathrm{E}[X^3] + 3\mathrm{E}[X^2Y] + 3\mathrm{E}[XY^2] + \mathrm{E}[Y^3]}{(\mathrm{Var}[X+Y])^{3/2}}$$

$$= \frac{\mathrm{E}[X^3] + 3\mathrm{E}[X^2]\cdot 0 + 3\cdot 0\cdot \mathrm{E}[Y^2] + \mathrm{E}[Y^3]}{2^{3/2}(\mathrm{Var}[X])^{3/2}}$$

$$= \frac{2\mathrm{E}[X^3]}{2^{3/2}\mathrm{SD}[X])^3} = \frac{\mathrm{Sk}(X)}{\sqrt{2}}$$

so (iii) is true and the answer is $\boxed{D}$

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12. $\mu = 0\frac{12}{100} + 100\frac{38}{100} + 200\frac{26}{100} + 300\frac{12}{100} + 400\frac{9}{100} + 500\frac{1}{100} + 600\frac{2}{100} = 179$ and $\mu_2' = 0^2\frac{12}{100} + 100^2\frac{38}{100} + 200^2\frac{26}{100} + 300^2\frac{12}{100} + 400^2\frac{9}{100} + 500^2\frac{1}{100} + 600^2\frac{2}{100} = 49{,}100$ so the empirical standard deviation is $\sqrt{49{,}100 - 179^2} = 130.6$ and the CV is $130.6/179 = \boxed{0.73}$

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13. $\mu = \dfrac{4+1+3+2+15}{5} = 5$, and the skewness is $\dfrac{\mathrm{E}[(X-\mu)^3]}{\sigma^3} = \dfrac{\frac{1}{5}\cdot 900}{\left(\frac{1}{5}130\right)^{3/2}} = \boxed{1.36}$

14.

$$\begin{aligned}
\mathrm{P}[N=2] &= \frac{P''(0)}{2} \\
P'(z) &= 5\left(0.3 + 0.2z + 0.5z^2\right)^4 \cdot (0.2 + z) \\
P''(z) &= 20\left(0.3 + 0.2z + 0.5z^2\right)^3 \cdot (0.2+z)^2 + 5\left(0.3 + 0.2z + 0.5z^2\right)^4 \\
P''(0) &= 20 \cdot 0.3^3 \cdot 0.2^2 + 5 \cdot 0.3^4 = 0.0621 \\
\mathrm{P}[N=2] &= \frac{0.0621}{2} = \boxed{0.031}
\end{aligned}$$

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15. See previous problem for $P'(z)$ and $P''(z)$.

$$\begin{aligned}
\mathrm{E}[N] &= P'(1) = 5 \cdot 1^4 \cdot 1.2 = 6 \\
\mathrm{E}[N(N-1)] &= P''(1) = 20 \cdot 1^3 \cdot 1.2^2 + 5 \cdot 1^4 = 33.8 \\
\mathrm{E}[N^2] - \mathrm{E}[N] &= \mathrm{E}[N^2] - 6 = 33.8 \Rightarrow \mathrm{E}[N^2] = 39.8 \\
\mathrm{Var}[N] &= 39.8 - 6^2 = \boxed{3.8}
\end{aligned}$$

16.

$$\begin{aligned}
\mathrm{P}[N=3] &= \frac{P'''(0)}{3!} \\
P'(z) &= \frac{1}{4}(1-z)^{-3/4} \\
P''(z) &= \frac{1}{4}\cdot\frac{3}{4}\cdot(1-z)^{-7/4} \\
P'''(z) &= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{7}{4}\cdot(1-z)^{-11/4} \\
P'''(0) &= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{7}{4} = \frac{21}{64} \\
\mathrm{P}[N=3] &= \frac{21}{64}\cdot\frac{1}{6} = \boxed{\frac{7}{128}}
\end{aligned}$$

17. $P'(1) = 2 = \mathrm{E}[X]$, and $P''(1) = \mathrm{E}[X(X-1)] = 6$. Expanding the second equation gives us $\mathrm{E}[X^2 - X] = \mathrm{E}[X^2] - \mathrm{E}[X] = \mathrm{E}[X^2] - 2 = 6$, so $\mathrm{E}[X^2] = 8$ and $\mathrm{Var}(X) = 8 - 2^2 = \boxed{4}$

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18. $M'(t) = (4t-5)e^{2t^2-5t}$ so $\mathrm{E}X = M'(0) = -5e^0 = -5$.
$M''(t) = 4e^{2t^2-5t} + (4t-5)^2 e^{2t^2-5t}$ so $\mathrm{E}[X^2] = 4e^0 + (-5)^2 e^0 = 4 + 5^2$.
This gives $\mathrm{Var}(X) = (4+5^2) - (-5)^2 = \boxed{4}$

Remark: $M_X(t)$ is the MGF of a normal random variable with mean -5 and variance $2 \cdot 2 = 4$, but you don't need to know that for the exam.

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19. $P'(z) = 3(4-3z)^{-2}$ so $\mathrm{E}[X] = P'(1) = 3$.
$P''(z) = 18(4-3z)^{-3}$ so $\mathrm{E}[X(X-1)] = P''(1) = 18$.
Expanding, we get $\mathrm{E}[X(X-1)] = \mathrm{E}[X^2] - \mathrm{E}[X]$ so $18 = \mathrm{E}[X^2] - 3$ and $\mathrm{E}[X^2] = \boxed{21}$

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20. There are 3 cases in which $X > Y$, namely $(2,1), (3,1)$ and $(3,2)$. Summing their probabilities gives $0.06 + 0.12 + 0.08 = \boxed{0.26}$

21.

$$\begin{aligned} \mathrm{P}[Y=1] = 0.3 \quad \mathrm{P}[Y=2] &= 0.4 \quad \mathrm{P}[Y=3] = 0.3 \\ \mathrm{E}[Y] &= 0.3 \cdot 1 + 0.4 \cdot 2 + 0.3 \cdot 3 = 2 \\ \mathrm{Var}[Y] &= 0.3 \cdot (1-2)^2 + 0.4 \cdot (2-2)^2 + 0.3 \cdot (3-2)^2 = 0.6 \\ \mathrm{CV}[Y] &= \frac{\sqrt{0.6}}{2} = \boxed{0.387} \end{aligned}$$

22.

$$\begin{aligned} \mathrm{P}[X > 0] &= 1 - 0.2 = 0.8 \\ \mathrm{P}[Y = 1 \mid X > 0] &= \frac{0.1}{0.8} \\ \mathrm{P}[Y = 2 \mid X > 0] &= \frac{0.4}{0.8} \\ \mathrm{P}[Y = 3 \mid X > 0] &= \frac{0.3}{0.8} \\ \mathrm{E}[Y \mid X > 0] &= 2.25 \\ \mathrm{Var}[Y \mid X > 0] &= \frac{7}{16} \\ \mathrm{CV}[Y \mid X > 0] &= \frac{\sqrt{7/16}}{2.25} = \boxed{0.294} \end{aligned}$$

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23. If N had a fixed, known value it would be easy, so we want to use double expectation. $\mathrm{E}[H] = \mathrm{E}[\mathrm{E}[H \mid N]] = \mathrm{E}[N/2] = (7/2)/2 = \boxed{7/4}$

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24. Using the conditional variance formula,

$$\mathrm{Var}(X) = \mathrm{E}[\mathrm{Var}(X \mid Y = y)] + \mathrm{Var}[\mathrm{E}(X \mid Y = y)]$$

$$= \mathrm{E}[2] + \mathrm{Var}[3Y] = 2 + 3^2\mathrm{Var}(Y)$$
$$= 2 + 9 \cdot 1^2 = \boxed{11}$$

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25. Since X and Y are independent, $\mathrm{Cov}(X,Y) = 0$ and $\mathrm{Var}[aX+bY] = a^2\mathrm{Var}[X]+b^2\mathrm{Var}[Y]$. Applying that with $a = b = 1$ gives (i), and with $a = 1, b = -1$ gives (ii), so those are true.
(iii) is false since $a^2\mathrm{Var}[X] = a^2\mathrm{E}[X^2] - a^2(\mathrm{E}[X])^2$, i.e., they are missing the square on the second term. So the answer is $\boxed{B}$

26.

$$F(x) = 0.4 \cdot \left[1 - \frac{1}{1+(x/10)^3}\right] + 0.6 \cdot \left[1 - \left(\frac{1}{1+(x/10)^3}\right)^2\right]$$
$$0.5 = 1 - 0.4u - 0.6u^2 \quad \text{where } u = \frac{1}{1+(x/10)^3}$$
$$0.6u^2 + 0.4u - 0.5 = 0$$
$$u = \frac{1}{1+(x/10)^3} = 0.638$$
$$1 + \left(\frac{x}{10}\right)^3 = 1.566$$
$$x = \boxed{8.27}$$

27.

$$\mathrm{Var}[X] = \mathrm{E}[\mathrm{Var}[X \mid \text{case}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{case}]]$$
$$\mathrm{E}[\mathrm{Var}[X \mid \text{case}]] = 0.5 \cdot 20 + 0.3 \cdot 50 + 0.2 \cdot 100 = 45$$
$$\mathrm{Var}[\mathrm{E}[X \mid \text{case}]] = 0.5 \cdot 10^2 + 0.3 \cdot 20^2 + 0.2 \cdot 40^2 - (0.5 \cdot 10 + 0.3 \cdot 20 + 0.2 \cdot 40)^2$$
$$= 129$$
$$\mathrm{Var}[X] = 45 + 129 = \boxed{174}$$

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28. Let p denote the proportion of low risk customers.

$$14 = 10p + 20(1-p)$$
$$p = 0.6$$
$$\mathrm{Var}[X] = \mathrm{E}[\mathrm{Var}[X \mid \text{Class}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{Class}]]$$
$$62 = 0.6 \cdot \mathrm{Var}[X \mid \text{Low}] + 0.4 \cdot 50 + 0.4 \cdot 0.6 \cdot (20-10)^2$$
$$\mathrm{Var}[X \mid \text{Low}] = \boxed{30}$$

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29. Here, we have the sum of 10 independent random variables, 6 with variance 50, and 4 with variance 20. For independent variables, the variance of the sum is the sum of the variances, and we have

$$6 \cdot 50 + 4 \cdot 20 = \boxed{380}$$

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30. Z is 1 if we chose the gold coin, and it lands with the 1 showing, so $\mathrm{P}[Z=1] = 1/4$. W cannot be 1 as the smallest it can be is 1.5 (when the gold coin is 1 and the silver is 2), so $\mathrm{P}[W=1] = 0$ and $a = 1/4$.

In fact, Z is uniform on $\{1,2,3,4\}$, so $\mathrm{Var}[Z] = 1.25$. Each coin individually has variance $2^2 \cdot 0.5^2 = 1$, so since $W = (\text{gold} + \text{silver})/2$, $\mathrm{Var}[W] = \frac{1}{4} \cdot (\mathrm{Var}[\text{gold}] + \mathrm{Var}[\text{silver}]) = 1/4(1+1) = 1/2$ and $b = 1.25 - 0.5 = 0.75$ giving us answer choice $\boxed{E}$