The Infinite Actuary Exam 4/C Online Seminar
A.1. Probability Review Solutions
Last updated December 3, 2015

1. [3-CAS.F03.17] Losses have an Inverse Exponential distribution. The mode is 10,000. Calculate the median.

   A. Less than 10,000
   B. At least 10,000, but less than 15,000
   C. At least 15,000, but less than 20,000
   D. At least 20,000, but less than 25,000
   E. At least 25,000

From the tables, the mode is $\theta/2 = 10{,}000$, so $\theta = 20{,}000$.
To find the median, set $0.5 = F(x) = e^{-20{,}000/x}$ so $x = -20,000/\ln(0.5) = \boxed{28{,}854}$

Alternatively, the median is $\mathrm{VaR}_{0.5}(X) = \theta(-\ln 0.5)^{-1} = 28{,}854$

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2. [3-CAS.F03.19] For a loss distribution where $x \geq 2$, you are given:

   (i) The hazard rate function: $h(x) = \dfrac{z^2}{2x}$, for $x \geq 2$
   (ii) A value of the distribution function: $F(5) = 0.84$.

   Calculate z.

   A. 2 B. 3 C. 4 D. 5 E. 6

$$H(5) = \int_0^5 h(x)dx = \int_0^2 0\,dx + \int_2^5 \frac{z^2}{2x}dx = \frac{z^2}{2}\ln(5) - \frac{z^2}{2}\ln(2) = \frac{z^2}{2}\ln(2.5)$$
$F(5) = 0.84 \Rightarrow S(5) = 1 - 0.84 = 0.16 = e^{-H(5)}$ and $H(5) = -\ln(0.16) = \dfrac{z^2}{2}\ln(2.5)$ so $\boxed{z = 2}$

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3. The density of Y is proportional to y^2 for $0 < y < 3$, and is 0 otherwise. Find the 80th percentile of Y.

   A. 0.9 B. 1.3 C. 1.8 D. 2.3 E. 2.8

$f(y) = cy^2, 0 < y < 3$, so $1 = \displaystyle\int_0^3 cy^2 dy = c\left.\frac{y^3}{3}\right|_0^3 = 9c$.
That gives us $c = 1/9$, and for the 80th percentile, $0.8 = \displaystyle\int_0^t \frac{1}{9}y^2 dy = \frac{t^3}{27}$ so $\boxed{t = 2.78}$

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4. A nonnegative random variable has a hazard rate function of $h(x) = A + e^{2x}, x \geq 0$. You are also given $S(0.4) = 0.5$.

Determine the value of A.

A. Less than 0.5
B. At least 0.5, but less than 1.0
C. At least 1.0, but less than 1.5
D. At least 1.5, but less than 2.0
E. At least 2.0

$$\begin{aligned} S(t) &= e^{-H(t)} \\ -\ln S(t) = H(t) &= \int_0^{0.4} A + e^{2x} dx \\ -\ln(0.5) &= 0.4A + \frac{1}{2}\left(e^{0.8} - 1\right) \\ A &= \boxed{0.2} \end{aligned}$$

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5. A nonnegative random variable X has a hazard rate function of $h(x) = \frac{3}{x}$ for $x > 2$ and 0 otherwise. Find E$[X]$.

A. 1 B. 2 C. 3 D. 6 E. 9

$$\begin{aligned} H(x) &= \int_0^2 0\,dt + \int_2^x \frac{3}{t} dt = 3\ln\left(\frac{x}{2}\right) \quad x > 2 \\ S(x) &= e^{-H(x)} = \begin{cases} 1 & x \leq 2 \\ \frac{8}{x^3} & x > 2 \end{cases} \\ \mathrm{E}[X] &= \int_0^2 1\,dx + \int_2^\infty \frac{8}{x^3} dx = 2 + \frac{4}{2^2} = \boxed{3} \\ \text{or: } f(x) &= -S'(x) = \frac{24}{x^4} \quad x > 2 \\ \mathrm{E}[X] &= \int_2^\infty x \cdot \frac{24}{x^4} dx = \int_2^\infty \frac{24}{x^3} dx = \frac{12}{2^2} = \boxed{3} \end{aligned}$$

Or: $X \sim$ single parameter Pareto$(\alpha = 3, \theta = 2)$ and $\mathrm{E}[X] = \dfrac{\alpha\theta}{\alpha - 1} = \dfrac{3 \cdot 2}{2} = 3$

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6. The survival function of X is $2 - x$ for $1 < x < 2$. Find E$[X]$.

A. 1/2 B. 5/4 C. 4/3 D. 3/2 E. 2

$$\begin{aligned} \mathrm{E}[X] &= \int_0^\infty S(x)\,dx = \int_0^1 1\,dx + \int_1^2 (2 - x) dx \\ &= 1 + 2 - \frac{2^2 - 1^2}{2} = \boxed{1.5} \end{aligned}$$

or: $f(x) = -S'(x) = 1 \quad 1 < x < 2$

$$\mathrm{E}[X] = \int_1^2 x \cdot 1 \, dx = \frac{2^2 - 1^2}{2} = \boxed{1.5}$$

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7. The survival function of X is $\dfrac{4 - x^2}{3}$ for $1 < x < 2$. Find Var$[X]$.

A. 0.080 B. 0.081 C. 0.082 D. 0.083 E. 0.084

$$f(x) = -S'(x) = \frac{2x}{3} \quad 1 < x < 2$$

$$\mathrm{E}[X] = \int_1^2 x \cdot \frac{2x}{3} \, dx = \frac{2}{9}\left(2^3 - 1^3\right) = \frac{14}{9}$$

$$\mathrm{E}[X^2] = \int_1^2 x^2 \cdot \frac{2x}{3} \, dx = \frac{2}{12}\left(2^4 - 1^4\right) = \frac{5}{2}$$

$$\mathrm{Var}[X] = \frac{5}{2} - \left(\frac{14}{9}\right)^2 = \frac{13}{162} = \boxed{0.080}$$

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8. [4.S01.3] You are given the following times of first claim for five randomly selected auto insurance policies observed from time $t = 0$:

1 2 3 4 5

Calculate the kurtosis of this sample.

A. 0.0 B. 0.5 C. 1.7 D. 3.4 E. 6.8

$\sigma^2 = \dfrac{(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2}{5} = 2$ and $\mu_4 = \dfrac{(-2)^4 + (-1)^4 + 0^4 + 1^4 + 2^4}{5} = \dfrac{34}{5}$

so the kurtosis is $(34/5)/4 = \boxed{1.7}$

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9. [3-CAS.F04.28] A large retailer of computers issues a warranty with each computer it sells. The warranty covers any cost to repair or replace a defective computer within 30 days of purchase. 40% of all claims are easily resolved and do not involve any cost to replace or repair. If a claim involves a cost to replace or repair, the claim size is distributed as a Weibull with parameters $\tau = \frac{1}{2}$ and $\theta = 30$.

   Which of the following statements are true?

   (i) The expected cost of a claim is \$60.

   (ii) The survival function at \$60 is 0.243.

   (iii) The hazard rate at \$60 is 0.012.

   A. (i) only.
   B. (ii) only.
   C. (iii) only.
   D. (i) and (ii) only.
   E. (ii) and (iii) only.

   From the exam tables, the mean of a Weibull distribution is $\theta\Gamma\left(1+\frac{1}{\tau}\right)$. To evaluate that, we will use the fact that $\Gamma(x) = (x-1)!$ when x is an integer. Since only 60% of claims result in a payment, we have $\text{E[cost]} = 0.6 \cdot \text{E[Weibull]} = 0.6 \cdot 30(3-1)! = 36$ so (i) is false and we can eliminate A and D.

   The survival function at 60 is 0.6 times the survival function of a Weibull at 60, giving us $0.6\left[e^{-(60/30)^{1/2}}\right] = 0.6 \cdot 0.243 = 0.146$ so (ii) is false and we can eliminate B and E.

   We now know the answer is $\boxed{C}$ (only choice left!) but let's confirm that (iii) holds:

   $$h(60) = \frac{f(60)}{S(60)} = \frac{0.6\left[\frac{1}{2}\left(\frac{60}{30}\right)^{1/2}\frac{1}{60}e^{-\sqrt{2}}\right]}{0.6e^{-\sqrt{2}}} = 0.01179.$$

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10. [C.F06.3] You are given a random sample of 10 claims consisting of two claims of 400, seven claims of 800, and one claim of 1600. Determine the empirical skewness coefficient.

    A. Less than 1.0
    B. At least 1.0, but less than 1.5
    C. At least 1.5, but less than 2.0
    D. At least 2.0, but less than 2.5
    E. At least 2.5

$$\begin{aligned} \mu &= \frac{1}{10}\left[2 \cdot 400 + 7 \cdot 800 + 1600\right] = 800 \\ \sigma^2 &= (0.2)(-400)^2 + (0.1)(800)^2 = 96{,}000 \\ \mu_3 &= (0.2)(-400)^3 + (0.1)(800)^3 = 38{,}400{,}000 \end{aligned}$$

The skewness is thus $38{,}400{,}000/(96{,}000)^{3/2} = \boxed{1.29}$

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11. [4B.S95.28] You are given the following:

- For any random variable X with finite first three moments, the skewness of the distribution of X is denoted $\mathrm{Sk}(X)$.
- X and Y are independent, identically distributed random variables with mean 0 and finite second and third moments.

Which of the following statements must be true?

(i) $2\mathrm{Sk}(X) = \mathrm{Sk}(2X)$

(ii) $-\mathrm{Sk}(Y) = \mathrm{Sk}(-Y)$

(iii) $|\mathrm{Sk}(X)| \geq |\mathrm{Sk}(X+Y)|$

A. (ii) only B. (iii) only C. (i) and (ii) only D. (ii) and (iii) only E. None of A, B, C, or D

$\mathrm{Sk}(cX) = \mathrm{Sk}(X)$ for $c > 0$, so (i) is false.
$\mathrm{SD}[-Y] = \mathrm{SD}[Y]$ but $\mathrm{E}\left[\left((-Y) - \mathrm{E}[-Y]\right)^3\right] = -\mathrm{E}\left[(Y - \mathrm{E}[Y])^3\right]$ so $\mathrm{Sk}(-Y) = -\mathrm{Sk}(Y)$ and (ii) is true.
Since X and Y are iid with mean 0,

$$\begin{aligned}
\mathrm{Sk}(X+Y) &= \frac{\mathrm{E}\left[(X+Y-0-0)^3\right]}{(\mathrm{SD}[X+Y])^3} \\
&= \frac{\mathrm{E}[X^3] + 3\mathrm{E}[X^2Y] + 3\mathrm{E}[XY^2] + \mathrm{E}[Y^3]}{(\mathrm{Var}[X+Y])^{3/2}} \\
&= \frac{\mathrm{E}[X^3] + 3\mathrm{E}[X^2]\cdot 0 + 3\cdot 0\cdot \mathrm{E}[Y^2] + \mathrm{E}[Y^3]}{2^{3/2}(\mathrm{Var}[X])^{3/2}} \\
&= \frac{2\mathrm{E}[X^3]}{2^{3/2}\mathrm{SD}[X])^3} = \frac{\mathrm{Sk}(X)}{\sqrt{2}}
\end{aligned}$$

so (iii) is true and the answer is $\boxed{D}$

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12. You observe the following losses:

Loss amount	0	100	200	300	400	500	600
Number of losses	12	38	26	12	9	1	2

Calculate the empirical coefficient of variation for the loss data.

A. 0.01 B. 0.73 C. 1.37 D. 10.58 E. 50.79

$\mu = 0\frac{12}{100} + 100\frac{38}{100} + 200\frac{26}{100} + 300\frac{12}{100} + 400\frac{9}{100} + 500\frac{1}{100} + 600\frac{2}{100} = 179$
and $\mu_2' = 0^2\frac{12}{100} + 100^2\frac{38}{100} + 200^2\frac{26}{100} + 300^2\frac{12}{100} + 400^2\frac{9}{100} + 500^2\frac{1}{100} + 600^2\frac{2}{100} = 49{,}100$ so the empirical standard deviation is $\sqrt{49{,}100 - 179^2} = 130.6$ and the CV is $130.6/179 = \boxed{0.73}$

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13. [3-CAS.S04.28] A pizza delivery company has purchased an automobile liability policy for its delivery drivers from the same insurance company for the past five years. The number of claims filed by the pizza delivery company as the result of at-fault accidents caused by its drivers is shown below:

Year	Claims
2002	4
2001	1
2000	3
1999	2
1998	15

Calculate the skewness of the empirical distribution of the number of claims per year.

A. Less than 0.50
B. At least 0.50, but less than 0.75
C. At least 0.75, but less than 1.00
D. At least 1.00, but less than 1.25
E. At least 1.25

$\mu = \dfrac{4+1+3+2+15}{5} = 5$, and the skewness is $\dfrac{\mathrm{E}[(X-\mu)^3]}{\sigma^3} = \dfrac{\frac{1}{5}\cdot 900}{\left(\frac{1}{5}130\right)^{3/2}} = \boxed{1.36}$

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14. A discrete distribution N has probability generating function

$$P_N(z) = (0.3 + 0.2z + 0.5z^2)^5$$

Find $\mathrm{P}[N = 2]$.

A. 0.023　B. 0.031　C. 0.046　D. 0.055　E. 0.062

$$\begin{aligned}
\mathrm{P}[N=2] &= \frac{P''(0)}{2} \\
P'(z) &= 5\left(0.3+0.2z+0.5z^2\right)^4 \cdot (0.2+z) \\
P''(z) &= 20\left(0.3+0.2z+0.5z^2\right)^3 \cdot (0.2+z)^2 + 5\left(0.3+0.2z+0.5z^2\right)^4 \\
P''(0) &= 20 \cdot 0.3^3 \cdot 0.2^2 + 5 \cdot 0.3^4 = 0.0621 \\
\mathrm{P}[N=2] &= \frac{0.0621}{2} = \boxed{0.031}
\end{aligned}$$

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15. A discrete distribution N has probability generating function

$$P_N(z) = (0.3 + 0.2z + 0.5z^2)^5$$

Find $\mathrm{Var}[N]$.

A. 1.9 B. 3.8 C. 7.6 D. 33.8 E. 39.8

See previous problem for $P'(z)$ and $P''(z)$.

$$\begin{aligned}
\mathrm{E}[N] &= P'(1) = 5 \cdot 1^4 \cdot 1.2 = 6 \\
\mathrm{E}[N(N-1)] &= P''(1) = 20 \cdot 1^3 \cdot 1.2^2 + 5 \cdot 1^4 = 33.8 \\
\mathrm{E}[N^2] - \mathrm{E}[N] &= \mathrm{E}[N^2] - 6 = 33.8 \Rightarrow \mathrm{E}[N^2] = 39.8 \\
\mathrm{Var}[N] &= 39.8 - 6^2 = \boxed{3.8}
\end{aligned}$$

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16. An actuary models the number of losses using a distribution with probability generating function

$$P(z) = 1 - (1-z)^{1/4}, \quad z < 1$$

According to the model, what is the probability of having exactly 3 losses?

A. $\frac{3}{64}$ B. $\frac{7}{128}$ C. $\frac{3}{16}$ D. $\frac{7}{32}$ E. $\frac{21}{64}$

$$\begin{aligned}
\mathrm{P}[N=3] &= \frac{P'''(0)}{3!} \\
P'(z) &= \frac{1}{4}(1-z)^{-3/4} \\
P''(z) &= \frac{1}{4} \cdot \frac{3}{4} \cdot (1-z)^{-7/4} \\
P'''(z) &= \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{7}{4} \cdot (1-z)^{-11/4} \\
P'''(0) &= \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{7}{4} = \frac{21}{64} \\
\mathrm{P}[N=3] &= \frac{21}{64} \cdot \frac{1}{6} = \boxed{\frac{7}{128}}
\end{aligned}$$

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17. [3.F06.25] You are given the following information about the probability generating function for a discrete distribution:

$$P'(1) = 2 \qquad P''(1) = 6$$

Calculate the variance of the distribution.

A. Less than 1.5
B. At least 1.5, but less than 2.5
C. At least 2.5, but less than 3.5
D. At least 3.5, but less than 4.5
E. At least 4.5

$P'(1) = 2 = \mathrm{E}[X]$, and $P''(1) = \mathrm{E}[X(X-1)] = 6$. Expanding the second equation gives us $\mathrm{E}[X^2 - X] = \mathrm{E}[X^2] - \mathrm{E}[X] = \mathrm{E}[X^2] - 2 = 6$, so $\mathrm{E}[X^2] = 8$ and $\mathrm{Var}(X) = 8 - 2^2 = \boxed{4}$

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18. The moment generating function of X is $M_X(t) = e^{2t^2-5t}$. Find $\mathrm{Var}[X]$.

A. 1 B. 2 C. 3 D. 4 E. 5

$M'(t) = (4t-5)e^{2t^2-5t}$ so $\mathrm{E}X = M'(0) = -5e^0 = -5$.
$M''(t) = 4e^{2t^2-5t} + (4t-5)^2 e^{2t^2-5t}$ so $\mathrm{E}[X^2] = 4e^0 + (-5)^2 e^0 = 4 + 5^2$.
This gives $\mathrm{Var}(X) = (4+5^2) - (-5)^2 = \boxed{4}$

Remark: $M_X(t)$ is the MGF of a normal random variable with mean -5 and variance $2 \cdot 2 = 4$, but you don't need to know that for the exam.

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19. You are given that the probability generating function of a random variable X is

$$P_X(z) = \frac{1}{4-3z}$$

Find the second raw moment of X.

A. 3 B. 9 C. 12 D. 18 E. 21

$P'(z) = 3(4-3z)^{-2}$ so $\mathrm{E}[X] = P'(1) = 3$.
$P''(z) = 18(4-3z)^{-3}$ so $\mathrm{E}[X(X-1)] = P''(1) = 18$.
Expanding, we get $\mathrm{E}[X(X-1)] = \mathrm{E}[X^2] - \mathrm{E}[X]$ so $18 = \mathrm{E}[X^2] - 3$ and $\mathrm{E}[X^2] = \boxed{21}$

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20. X and Y are discrete variables whose joint distribution $\mathrm{P}[X = x, Y = y] = p(x, y)$ is given by

$$\begin{array}{lll} p(1,1) = 0.12 & p(2,1) = 0.06 & p(3,1) = 0.12 \\ p(1,2) = 0.00 & p(2,2) = 0.12 & p(3,2) = 0.08 \\ p(1,3) = 0.20 & p(2,3) = 0.05 & p(3,3) = 0.15 \\ p(1,4) = 0.05 & p(2,4) = 0.02 & p(3,4) = 0.03 \end{array}$$

Find $\mathrm{P}[X > Y]$

A. 0.26 B. 0.30 C. 0.32 D. 0.35 E. 0.39

There are 3 cases in which $X > Y$, namely $(2,1), (3,1)$ and $(3,2)$. Summing their probabilities gives $0.06 + 0.12 + 0.08 = \boxed{0.26}$

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21. X and Y are discrete variables whose joint distribution $\mathrm{P}[X = x, Y = y] = p(x, y)$ is given by

$$\begin{array}{ll} p(0,1) = 0.2 & p(1,1) = 0.1 \\ p(1,2) = 0.3 & p(2,2) = 0.1 \\ p(2,3) = 0.1 & p(3,3) = 0.2 \end{array}$$

Find the coefficient of variation of Y.

A. 0.30 B. 0.39 C. 0.60 D. 2.58 E. 3.33

$$\mathrm{P}[Y = 1] = 0.3 \quad \mathrm{P}[Y = 2] = 0.4 \quad \mathrm{P}[Y = 3] = 0.3$$
$$\mathrm{E}[Y] = 0.3 \cdot 1 + 0.4 \cdot 2 + 0.3 \cdot 3 = 2$$
$$\mathrm{Var}[Y] = 0.3 \cdot (1-2)^2 + 0.4 \cdot (2-2)^2 + 0.3 \cdot (3-2)^2 = 0.6$$
$$\mathrm{CV}[Y] = \frac{\sqrt{0.6}}{2} = \boxed{0.387}$$

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22. X and Y are discrete variables whose joint distribution $\mathrm{P}[X = x, Y = y] = p(x, y)$ is given by

$$\begin{array}{ll} p(0,1) = 0.2 & p(1,1) = 0.1 \\ p(1,2) = 0.3 & p(2,2) = 0.1 \\ p(2,3) = 0.1 & p(3,3) = 0.2 \end{array}$$

Find the coefficient of variation of Y conditioned on X being positive.

A. 0.29 B. 0.31 C. 0.33 D. 0.35 E. 0.37

$$\mathrm{P}[X > 0] = 1 - 0.2 = 0.8$$
$$\mathrm{P}[Y = 1 \mid X > 0] = \frac{0.1}{0.8}$$

$$\begin{aligned} P[Y=2 \mid X>0] &= \frac{0.4}{0.8} \\ P[Y=3 \mid X>0] &= \frac{0.3}{0.8} \\ E[Y \mid X>0] &= 2.25 \\ \mathrm{Var}[Y \mid X>0] &= \frac{7}{16} \\ \mathrm{CV}[Y \mid X>0] &= \frac{\sqrt{7/16}}{2.25} = \boxed{0.294} \end{aligned}$$

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23. Let N be the value rolled by a fair six-sided die. Suppose that I then flip N independent fair coins. What is the expected number of heads?

A. 1/4 B. 3/4 C. 3/2 D. 7/4 E. 7/2

If N had a fixed, known value it would be easy, so we want to use double expectation. $E[H] = E[E[H \mid N]] = E[N/2] = (7/2)/2 = \boxed{7/4}$

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24. (Based on [110.F86.41]) Let $E[X \mid Y=y] = 3y$, $\mathrm{Var}[X \mid Y=y] = 2$ and let Y be an exponential random variable with mean 1.

What is $\mathrm{Var}[X]$?

A. 3 B. 5 C. 9 D. 11 E. 20

Using the conditional variance formula,

$$\begin{aligned} \mathrm{Var}(X) &= E[\mathrm{Var}(X \mid Y=y)] + \mathrm{Var}[E(X \mid Y=y)] \\ &= E[2] + \mathrm{Var}[3Y] = 2 + 3^2\mathrm{Var}(Y) \\ &= 2 + 9 \cdot 1^2 = \boxed{11} \end{aligned}$$

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25. [4B.S93.9] If X and Y are independent random variables, which of the following statements are true?

(i) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
(ii) $\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$
(iii) $\mathrm{Var}(aX+bY) = a^2E[X^2] - a(E[X])^2 + b^2E[Y^2] - b(E[Y])^2$

A. (i) only B. (i) and (ii) only C. (i) and (iii) only D. (ii) and (iii) only E. (i), (ii) and (iii)

Since X and Y are independent, $\mathrm{Cov}(X,Y) = 0$ and $\mathrm{Var}[aX+bY] = a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y]$. Applying that

with $a = b = 1$ gives (i), and with $a = 1, b = -1$ gives (ii), so those are true.
(iii) is false since $a^2\text{Var}[X] = a^2\text{E}[X^2] - a^2(\text{E}[X])^2$, i.e., they are missing the square on the second term.
So the answer is $\boxed{B}$

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26. An insurer groups its policyholders into two groups. Losses from members of group A have a Burr distribution with $\theta = 10$, $\gamma = 3$ and $\alpha = 1$, while losses from members of group B have a Burr distribution with $\theta = 10, \gamma = 3$ and $\alpha = 2$.

    If 40% of losses are from policyholders from group A, find the median loss amount

    A. 8.07 B. 8.17 C. 8.27 D. 8.37 E. 8.47

$$F(x) = 0.4 \cdot \left[1 - \frac{1}{1 + (x/10)^3}\right] + 0.6 \cdot \left[1 - \left(\frac{1}{1 + (x/10)^3}\right)^2\right]$$
$$0.5 = 1 - 0.4u - 0.6u^2 \quad \text{where } u = \frac{1}{1 + (x/10)^3}$$
$$0.6u^2 + 0.4u - 0.5 = 0$$
$$u = \frac{1}{1 + (x/10)^3} = 0.638$$
$$1 + \left(\frac{x}{10}\right)^3 = 1.566$$
$$x = \boxed{8.27}$$

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27. An insurance company has 3 types of customers, whose annual losses are described below. Find the variance of the annual loss of a randomly chosen customer.

Type	Proportion of Customers	Average Annual Loss	Variance of Annual Losses
Low	50%	10	20
Med	30%	20	50
High	20%	40	100

    A. 45 B. 80 C. 123 D. 174 E. 219

$$\text{Var}[X] = \text{E}[\text{Var}[X \mid \text{case}]] + \text{Var}[\text{E}[X \mid \text{case}]]$$
$$\text{E}[\text{Var}[X \mid \text{case}]] = 0.5 \cdot 20 + 0.3 \cdot 50 + 0.2 \cdot 100 = 45$$
$$\text{Var}[\text{E}[X \mid \text{case}]] = 0.5 \cdot 10^2 + 0.3 \cdot 20^2 + 0.2 \cdot 40^2 - (0.5 \cdot 10 + 0.3 \cdot 20 + 0.2 \cdot 40)^2$$
$$= 129$$
$$\text{Var}[X] = 45 + 129 = \boxed{174}$$

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28. An insurance company has two types of customers: high risk customers, whose loss amounts have mean 20 and variance 50, and low risk customers, whose losses have mean 10. The expected amount of a randomly chosen loss is 14, and the variance of a randomly selected loss is 62. Find the variance of a loss from a low risk customer.

A. 12 B. 18 C. 30 D. 48 E. 70

Let p denote the proportion of low risk customers.

$$\begin{aligned}
14 &= 10p + 20(1-p) \\
p &= 0.6 \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \text{Class}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{Class}]] \\
62 &= 0.6 \cdot \mathrm{Var}[X \mid \text{Low}] + 0.4 \cdot 50 + 0.4 \cdot 0.6 \cdot (20-10)^2 \\
\mathrm{Var}[X \mid \text{Low}] &= \boxed{30}
\end{aligned}$$

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29. An insurance company has two types of customers: high risk customers, whose annual loss amounts have mean 20 and variance 50, and low risk customers, whose annual losses have mean 10 and variance 20.

The loss amounts of the different customers are all independent. In a group with 6 high risk and 4 low risk customers, what is the variance of the total annual losses from the group?

A. 38 B. 62 C. 289 D. 380 E. 620

Here, we have the sum of 10 independent random variables, 6 with variance 50, and 4 with variance 20. For independent variables, the variance of the sum is the sum of the variances, and we have

$$6 \cdot 50 + 4 \cdot 20 = \boxed{380}$$

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30. I have two fair, but unusual coins. One is gold, and the two sides are marked "1" and "3," while the other is silver with its sides marked "2" and "4." Suppose that I flip both coins. Let W be the average of the two sides that land face up. Let Z be the value that is face up if I choose one of the two coins at random to look at.

Let $a = \mathrm{P}[Z = 1] - \mathrm{P}[W = 1]$, and let $b = \mathrm{Var}[Z] - \mathrm{Var}[W]$. Find a and b.

A. $a = 0, b = 0$
B. $a = 0, b = 0.25$
C. $a = 0.25, b = 0.25$
D. $a = 0.25, b = 0.5$
E. $a = 0.25, b = 0.75$

Z is 1 if we chose the gold coin, and it lands with the 1 showing, so $\mathrm{P}[Z = 1] = 1/4$. W cannot be 1 as the smallest it can be is 1.5 (when the gold coin is 1 and the silver is 2), so $\mathrm{P}[W = 1] = 0$ and $a = 1/4$.

In fact, Z is uniform on $\{1, 2, 3, 4\}$, so $\mathrm{Var}[Z] = 1.25$. Each coin individually has variance $2^2 \cdot 0.5^2 = 1$, so since $W = (\text{gold} + \text{silver})/2$, $\mathrm{Var}[W] = \frac{1}{4} \cdot (\mathrm{Var}[\text{gold}] + \mathrm{Var}[\text{silver}]) = 1/4(1+1) = 1/2$ and $b = 1.25 - 0.5 = 0.75$ giving us answer choice $\boxed{E}$