C.4 Bayesian Statistics

C.4.2 Poisson/Gamma Conjugate Prior

- Conjugate Priors
- Poisson/Gamma
- Exercises

Sources

This lesson comes from Hogg McKean & Craig (7th Ed.) section 11.2 and Wackerly sections 16.3

Conjugate Priors

$X \sim F(x \mid \theta) =$ **model distribution**

$\pi(\theta) =$ **prior distribution**

$\pi(\theta \mid X) =$ **posterior distribution** $= \dfrac{\text{Joint Density}(\theta, X)}{\text{Unconditional density}(X)}$

If $\pi(\theta \mid X)$ has the same form as $\pi(\theta)$ with different parameter values

$\longrightarrow \pi$ is a **conjugate prior.**

$N \sim \text{Poisson}(\lambda)$, $\lambda \sim \text{Gamma}(\alpha, \theta) \longrightarrow$ Poisson/Gamma

$N \sim \text{Binomial}(n, p)$, $p \sim \text{Beta}(a, b) \longrightarrow$ Binomial/Beta

$X \sim N(\theta, v)$,$\theta \sim N(\mu, a) \longrightarrow$ Normal/Normal

Poisson/Gamma

$N \sim \text{Poisson}(\lambda)$ = conditional distribution $\longrightarrow p_k = e^{-\lambda}\lambda^k/k!$

$\lambda \sim \text{Gamma}(\alpha, \theta) \longrightarrow \pi(\lambda) = \dfrac{\lambda^{\alpha-1}e^{-\lambda/\theta}}{\theta^\alpha(\alpha-1)!}$

Observe $N = k$.

$$\pi(\lambda \mid N = k) = \frac{\text{Joint Density}(\lambda, k)}{\text{Unconditional density}(k)} = \frac{\pi(\lambda)\mathrm{P}[N = k \mid \lambda]}{\mathrm{P}[N = k]}$$

$$= \frac{\left(\dfrac{\lambda^{\alpha-1}e^{-\lambda/\theta}}{\theta^\alpha(\alpha-1)!}\right)\left(\dfrac{e^{-\lambda}\lambda^k}{k!}\right)}{\mathrm{P}[N = k]} = C\,\lambda^{\alpha-1+k}e^{-\lambda(1/\theta+1)}$$

C doesn't depend on λ. Could find on chart or $\int \pi(\lambda \mid N = k)\, d\lambda = 1$.

But already $\pi(\lambda \mid N = k) \sim \text{Gamma}\left(\alpha' = \alpha + k, \dfrac{1}{\theta'} = \dfrac{1}{\theta} + 1\right)$.

Example

The annual number of claims for a policyholder follows a Poisson distribution with mean λ. The prior distribution of λ is Gamma with probability density function $f(\lambda) = \dfrac{(2\lambda)^5 e^{-2\lambda}}{24\lambda}$. An insured selected at random has $x_1 = 5$ claims during Year 1 and $x_2 = 3$ during year 2.

Find the posterior distribution of λ. $\longrightarrow$ Gamma$(\alpha = 5, \theta = 1/2)$

$$\pi(\lambda \mid \mathbb{X}) = \frac{\left(\dfrac{\lambda^{5-1} e^{-\lambda/(1/2)}}{\theta^5 (5-1)!}\right) \left(e^{-\lambda}\lambda^5/5!\right) \left(e^{-\lambda}\lambda^3/3!\right)}{\mathrm{P}[x_1 = 5, x_2 = 3]}$$

$$= C\,\lambda^{5-1+(5+3)} e^{-\lambda\left(\frac{1}{1/2}+2\right)}$$

$$\sim \text{Gamma}\left(\alpha' = 5 + (5+3), \frac{1}{\theta'} = \frac{1}{1/2} + 2\right)$$

$$\sim \text{Gamma}\left(\alpha' = \alpha + \text{ total claims }, \frac{1}{\theta'} = \frac{1}{\theta} + \text{ total years}\right)$$

Posterior mean claims

Recent exam question: Calculate the posterior mean claims.

Poisson/Gamma: the answer is the same as mean of the posterior.

$$\mathrm{E}[N \mid \mathbb{X}] = \mathrm{E}_{\pi(\lambda|\mathbb{X})}\left[\mathrm{E}\left[N \mid \lambda\right]\right] = \mathrm{E}_{\pi(\lambda|\mathbb{X})}\left[\lambda\right] = \alpha' \cdot \theta'.$$

In case you are asked for the posterior *variance* of claims:

For Poisson/Gamma, $N \sim$ Negative Binomial with $r = \alpha,\ \beta = \theta$.

$$\mathrm{E}[N] = r \cdot \beta = \alpha \cdot \theta \qquad \mathrm{Var}(N) = r \cdot \beta(1+\beta) = \alpha \cdot \theta(1+\theta).$$

Compare/Contrast Posterior mean/variance:

$$\mathrm{E}[\lambda] = \alpha' \cdot \theta', \qquad \mathrm{Var}(\lambda) = \alpha' \cdot \theta'^2$$

$$\mathrm{E}[N] = \alpha' \cdot \theta', \qquad \mathrm{Var}(N) = \alpha' \cdot \theta'(1+\theta')$$

The posterior distribution of N is called the **predictive distribution.**

The posterior mean of N is the **Bayesian credibility premium**.

Exercise 1

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$E[\lambda] = 2.5 = \alpha\theta$ $\quad$ $Var(\lambda) = \alpha\theta^2 = 1.25$

$(\alpha\theta)\theta$ $\quad$ $\theta = \frac{1.25}{2.5} = \frac{1}{2}$

$k = 1 \quad n = 1$ $\quad$ $2.5\theta = 1.25$ $\quad$ $\alpha = 5$

$$\text{Gamma}\left(\alpha' = 5 + 1 = 6,\ \frac{1}{\theta'} = \frac{1}{1/2} + 1 = 3\right)$$

$$\boxed{= \text{Gamma}(\alpha' = 6,\ \theta' = 1/3)}$$

Exercise 2

The annual number of claims for a policyholder follows a Poisson distribution with mean λ. The prior distribution of λ has density $\lambda e^{-\lambda}$ An insured selected at random has 1 claim during Year 1 and x claims in year 2. The posterior density of λ is found to be $13.5\lambda^2 e^{-3\lambda}$. How many claims were observed in Year 2?

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$\pi(\lambda) = \text{Gamma}(\alpha = 2, \theta = 1) \quad k = 1 + x \quad n = 2$

$\pi(\lambda \mid X) = \text{Gamma}\left(\alpha' = 2 + 1 + x,\ \frac{1}{\theta'} = \frac{1}{1} + 2 = 3\right)$

$\underline{\theta' = 1/3}$

$\alpha' = 3$

$\Rightarrow \boxed{x = 0}$