The Infinite Actuary Exam S Online Seminar
CAS S C.4.2 Problems with Solutions

1. First Pass:
The probability distribution function of claims per year for an individual risk is a Poisson distribution with parameter λ. The prior distribution of λ is a gamma distribution with density

$$f(\lambda) = \lambda e^{-\lambda}, \qquad 0 < \lambda < \infty.$$

Given an observation of one claim in a one-year period, what is the density of the posterior distribution of λ?

A. $e^{-\lambda}$ B. $\dfrac{\lambda^2}{2}e^{-\lambda}$ C. $4\lambda^2 e^{-2\lambda}$ D. $\lambda e^{-2\lambda}$ E. $4\lambda e^{-2\lambda}$

2. For a large number of health insurance policies the annual numbers of claims follow the Poisson distribution with mean λ, which varies by policy. The prior probability density function of λ is

$$h(\lambda) = \left(\frac{1}{2}\right)\lambda^2 e^{-\lambda}, \lambda > 0$$

In Year 1, four claims are reported for a policy.

Calculate the expected value of the posterior distribution of λ for this policy.

A. Less than 2.0
B. At least 2.0, but less than 3.0
C. At least 3.0, but less than 4.0
D. At least 4.0, but less than 5.0
E. At least 5.0

3. You are given:

- An individual automobile insured has an annual claim frequency distribution that follows a Poisson distribution with mean λ.
- λ follows a gamma distribution with parameters α and θ.
- The first actuary assumes that $\alpha = 1$ and $\theta = 1/6$.
- The second actuary assumes the same mean for the gamma distribution, but only half the variance.
- A total of one claim is observed for the insured over a three year period.
- Both actuaries determine the posterior mean for λ using their model assumptions.

Determine the ratio of the posterior mean that the first actuary calculates to the posterior mean that the second actuary calculates.

A. 3/4 B. 9/11 C. 10/9 D. 11/9 E. 4/3

4. The probability distribution function of claims per year for an individual risk is a Poisson distribution with parameter λ. The prior distribution of λ is a gamma distribution with density

$$f(\lambda) = \lambda e^{-\lambda}, \qquad 0 < \lambda < \infty.$$

Given an observation of one claim in a one-year period, what is the posterior probabilty that λ exceeds 2?

A. Less than 0.1
B. At least 0.1, but less than 0.2
C. At least 0.2, but less than 0.3
D. At least 0.3, but less than 0.4
E. At least 0.4

5. You are given the following information for an insurance policy:

- Monthly claim frequencies follow a Poisson process with parameter λ.
- The prior distribution of λ follows a gamma distribution with parameters $\alpha = 3$ and $\theta = 2$.
- In the first month, a policy had 27 claims.

Calculate the posterior mean monthly claims for this policy.

A. Less than 9
B. At least 9, but less than 13
C. At least 13, but less than 17
D. At least 17, but less than 21
E. At least 21

6. You are given the following information:

- $X_1, \ldots, X_5$ are a random sample from a Poisson distribution with parameter λ, where λ follows a gamma distribution with parameters $\alpha = 2$ and θ.
- The mean of this Poisson-gamma conjugate pair can be represented as a weighted average of the maximum likelihood estimator for the mean and the mean of the prior distribution.
- Let W_{MLE} be the weight assigned to the maximum likelihood estimator.
- The maximum likelihood estimate for the mean is 1.2.
- The variance of the prior gamma is 8.

Calculate W_{MLE}.

A. Less than 0.60
B. At least 0.60, but less than 0.70
C. At least 0.70, but less than 0.80
D. At least 0.80, but less than 0.90
E. At least 0.90

7. You are given:

- The number of claims per year for a policyholder follows a Poisson distribution with mean λ.
- The prior distribution for the mean, λ, follows a gamma distribution with mean 0.2 and variance 0.005.
- Last year, 60 annual policies produced a total of 12 claims.

Calculate the variance of the posterior distribution of λ.

A. Less than 0.0015
B. At least 0.0015, but less than 0.0025
C. At least 0.0025, but less than 0.0035
D. At least 0.0035, but less than 0.0045
E. At least 0.0045

8. You are given:

- Daily claim counts follow a Poisson distribution with mean λ.
- The prior distribution of λ has probability density function:

$$f(\lambda) = \frac{1}{5}e^{-\lambda/5}$$

- Three claims were observed on the first day.

Calculate the variance of the posterior distribution of λ.

A. Less than 1.0
B. At least 1.0, but less than 2.0
C. At least 2.0, but less than 3.0
D. At least 3.0, but less than 4.0
E. At least 4.0

9. The number of claims for an individual risk in a single year follows a Poisson distribution with parameter λ. The parameter λ has for a prior distribution the following gamma density function with parameters $\alpha = 1$ and $\theta = 2$:

$$f(\lambda) = \frac{1}{2}e^{-\frac{\lambda}{2}},\ \lambda > 0.$$

You are given that three claims arose in the first year.

Determine the posterior distribution of λ.

A. $\frac{1}{2}e^{-\frac{3}{2}\lambda}$ B. $\frac{1}{12}\lambda^3 e^{-\frac{1}{2}\lambda}$ C. $\frac{1}{4}\lambda^3 e^{-\frac{1}{2}\lambda}$ D. $\frac{27}{32}\lambda^3 e^{-\frac{3}{2}\lambda}$ E. $\frac{1}{12}\lambda^2 e^{-\frac{3}{2}\lambda}$

10. The number of claims for an individual risk in a single year follows a Poisson distribution with parameter λ. The parameter λ has for a prior distribution the following gamma density function with parameters $\alpha = 1$ and $\theta = 2$:

$$f(\lambda) = \frac{1}{2}e^{-\frac{\lambda}{2}},\ \lambda > 0.$$

You are given that three claims arose in the first year.

Determine the Bayesian estimate for the expected number of claims in the second year.

A. Less than 2.25
B. At least 2.25, but less than 2.50
C. At least 2.50, but less than 2.75
D. At least 2.75, but less than 3.00
E. At least 3.00

11. You are given the following:

- For an Individual risk In a population, the number of claims for a single exposure period follows a Poisson distribution with mean λ.
- For the population, λ is distributed according to an exponential distribution with mean 0.1;

$$g(\lambda) = 10e^{-10\lambda}, \lambda > 0$$

- An individual risk is selected at random from the population.
- After one exposure period, one claim has been observed.

Determine the density function of the posterior distribution of λ for the selected risk.

A. $11e^{-11\lambda}$ B. $10\lambda e^{-11\lambda}$ C. $121\lambda e^{-11\lambda}$ D. $\frac{1}{10}e^{-9\lambda}$ E. $\frac{11e^{-11\lambda}}{\lambda^2}$

12. You are given the following:

- The number of claims per year for a given risk follows a Poisson distribution with mean λ.
- The prior distribution of λ is assumed to be a gamma distribution with mean 1/2 and variance 1/8.

Determine the variance of the posterior distribution of λ if a total of 4 claims have been observed for this risk in a 2-year period.

A. $\frac{1}{16}$ B. $\frac{1}{8}$ C. $\frac{1}{6}$ D. $\frac{1}{2}$ E. 1

13. You are given the following:

r is a random variable that represents the number of claims for an individual risk and has the Poisson density function

$$f(r) = \frac{t^r e^{-t}}{r!}, r = 0, 1, 2, ...$$

The parameter t has a prior gamma distribution with density function

$$h(t) = 5e^{-5t}, t > 0$$

A portfolio consists of 100 independent risks, each having identical density functions.

In one year 10 claims are experienced by the portfolio.

Determine the Bayesian expected number of claims in the second year for the portfolio.

A. Less than 6
B. At least 6, but less than 8
C. At least 8, but less than 10
D. At least 10, but less than 12
E. At least 12

14. An individual automobile insured has a claim count distribution per policy period that follows a Poisson distribution with parameter λ. For the overall population of insureds, λ follows a distribution with density function

$$f(\lambda) = 5exp(-5\lambda), \lambda > 0$$

One insured is selected at random from the population and is observed to have a total of one claim during two policy periods.

Determine the expected number of claims that this same insured will have during the third policy period.

A. Less than 0.2
B. At least 0.2, but less than 0.4
C. At least 0.4, but less than 0.6
D. At least 0.6, but less than 0.8
E. At least 0.8

15. You are given the following:

- A portfolio consists of 1000 identical and independent risks.
- The number of claims for each risk follows a Poisson distribution with mean λ.
- Prior to the latest exposure period, λ is assumed to have a gamma distritbution, with parameiers $\alpha = 250$ and $\theta = 0.0005$.
- During the latest exposure period, the following loss experience is observed:

Number of Claims	Number of Risks
0	906
1	89
2	4
3	1
	1000

Determine the mean of the posterior distribution of λ.

A. Less than 0.11
B. At least 0.11, but less than 0.12
C. At least 0.12, but less than 0.13
D. At least 0.13, but less than 0.14
E. At least 0.14

16. You are given the following:

- A portfolio consists of 100 identical and independent risks.
- The number of claims per year for each risk follows a Poisson distribution with mean λ.
- The prior distribution of λ is assumed to be a gamma distribution with mean 0.25 and variance 0.0025.
- During the latest year, the following loss experience is observed:

Number of Claims	Number of Risks
0	80
1	17
2	3

Determine the variance of the posterior distribution of λ.

A. Less than 0.00075
B. At least 0.00075, but less than 0.00125
C. At least 0.00125, but less than 0.00175
D. At least 0.00175, but less than 0.00225
E. At least 0.00225

17. You are given:

- Annual claim counts follow a Poisson distribution with mean λ.
- The parameter λ has a prior distribution with probability density function:

$$f(\lambda) = \frac{1}{3}e^{-\lambda/3}, \lambda > 0$$

Two claims were observed during the first year.

Determine the variance of the posterior distribution of λ.

A. $\frac{9}{16}$ B. $\frac{27}{16}$ C. $\frac{9}{4}$ D. $\frac{16}{3}$ E. $\frac{27}{4}$

18. You are given:

- The number of claims per auto insured folows a Poisson disbibution with mean λ.
- The prior distribution for λ has the following probability density function:

$$f(\lambda) = \frac{(500\lambda)^{50} e^{-500\lambda}}{\lambda \Gamma(50)}$$

- A company observes the following claims experience:

	Year 1	Year 2
Number of claims	75	210
Number of autos insured	600	900

The company expects to insure 1100 autos in Year 3.

Determine the expected number of claims in Year 3.

A. 178 B. 184 C. 193 D. 209 E. 224

19. The annual number of claims for a policy follows the Poisson distribution with mean λ, which varies by policy. The prior probability density function of λ is

$$h(\lambda) = \lambda e^{-\lambda}, \quad \lambda > 0$$

In Year 1, three claims are reported for a policy.

Calculate the coefficient of variation (the standard deviation divided by the mean) of the posterior distribution of λ, for this policy.

A. Less than 0.5
B. At least 0.5, but less than 1.0
C. At least 1.0, but less than 1.5
D. At least 1.5, but less than 2.0
E. At least 2.0

20. You are given:

- The number of claims per year for a policyholder follows a Poisson distribution with mean λ.
- The mean λ follows a gamma distribution with parameters $\alpha = 3$ and $\theta = 10$.
- A policyholder had 1 claim last year.

Calculate the mean of the posterior distribution of λ.

A. Less than 2.0
B. At least 2.0, but less than 4.0
C. At least 4.0, but less than 6.0
D. At least 6.0, but less than 8.0
E. At least 8.0