1. B.2.1 Multi-Period Binomial Option Pricing
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   - Pricing European Options
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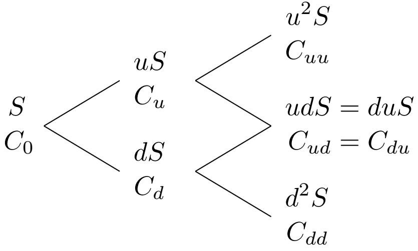
Multi-Period Binomials

To allow for more possible stock prices, we can combine individual binomial steps to create multi-period trees



Note that when u and d are constant, the tree will recombine. I.e., $udS = duS$

Binomial Models in the Limit

Let T be the option's expiration and n be the number of binomial steps between time 0 and time T. (Note that $h = T/n$.) Then as $n \to \infty$:

1. Forward, lognormal and CRR trees will all result in the same option price
2. S_T will be distributed lognormal

To solve multi-stage binomial option problems:

1. Start by computing the option payoffs at expiration at the far right of the tree
2. Work backwards through the tree (i.e., right to left) solving each individual binomial step in the tree for the binomial option price

Note that in recombining trees, p^* will remain constant throughout the tree; whereas, Δ and B will not

Thus, the risk-neutral pricing method is generally preferred for multi-period problems

Multi-Period Binomial Pricing Example

Example

Given $S_0 = 50$, $\delta = 0.1$, $\sigma = 0.2$, $h = 6$ months, and $r = 2\%$, use a forward tree to price an at-the-money European call option expiring in 1 year.

1. Solve for u and d

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = 1.1067, \quad d = e^{(0.02-0.1)0.5-0.2\sqrt{0.5}} = 0.8341$$

2. Complete stock tree

$$\begin{array}{ccccc} & & & & u^2S = 61.2440 \\ & & uS = 55.3371 & \nearrow & \\ S = 50 & \nearrow\!\!\!\searrow & & \searrow\!\!\!\nearrow & udS = 46.1588 \\ & & dS = 41.7042 & \searrow & \\ & & & & d^2S = 34.7844 \end{array}$$

Example *(continued)*

3. Calculate the call payoffs at expiration

$$\begin{array}{ccccc}
 & & & & 61.2440 \\
 & & 55.3371 & & C_{uu} = \max(0, 61.2440 - 50) = 11.244 \\
 & & C_u & & \\
50 & & & & 46.1588 \\
C_0 & & & & C_{ud} = \max(0, 46.1588 - 50) = 0 \\
 & & 41.7042 & & \\
 & & C_d & & 34.7844 \\
 & & & & C_{dd} = \max(0, 34.7844 - 50) = 0
\end{array}$$

4. Calculate p^*

$$p^* = \frac{e^{(r-\delta)h} - d}{u - d} = \frac{e^{(.02-.1)(.5)} - .8341}{1.1067 - .8341} = 0.4647$$

Example *(continued)*

5. Solve for C_u and C_d

$$C_u = e^{-rh}\left[p^* C_{uu} + (1-p^*)C_{ud}\right]$$
$$= e^{-.02(.5)}[(0.4647)(11.244) + (1-.4647)(0)] = 5.1731$$

$$C_d = e^{-rh}\left[p^* C_{ud} + (1-p^*)C_{dd}\right]$$
$$= e^{-.02(.5)}[0 + 0] = 0$$

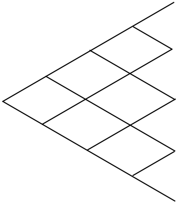
6. Solve for C_0

$$C_0 = e^{-rh}\left[p^* C_u + (1-p^*)C_d\right]$$
$$= e^{-.02(.5)}[(0.4647)(5.1731) + (1-.4647)(0)] = 2.38$$

Binomial Tree Probabilities

If you label the end nodes from $i = 0$ to n, the number of paths to reach the ith node in an n-period binomial tree is $\binom{n}{i}$

E.g., consider a tree with $n = 4$ periods



$i = 0$	1 path
$i = 1$	4 paths
$i = 2$	6 paths
$i = 3$	4 paths
$i = 4$	1 path

Let p^* be the risk-neutral probability of an up move, then the risk-neutral probability of reaching node i is:

$$(p^*)^{n-i}(1 - p^*)^i \binom{n}{i}$$

Multi-period Binomial Pricing of European Options

Since early exercise is not possible, we can price a European option by discounting the expected risk-neutral payoff at time T back to time 0 in a single step:

$$C = e^{-rT} \sum_{i=0}^{n} \left[(p^*)^{n-i} (1-p^*)^i \binom{n}{i} \max\left(0, u^{n-i} d^i S_0 - K\right) \right]$$

Applying to the previous example:

$$C = e^{-.02(1)} \left[(p^*)^2 (1)(11.244) + p^*(1-p^*)(2)(0) + (1-p^*)^2 (1)(0) \right]$$

$$= 2.38$$

Multi-Period Binomial Pricing of American Options

The above procedure will not work for American options (except an American call on a non-dividend paying stock)

To price American binomial options:

1. At every intermediate node, starting at the right, decide whether early exercise is optimal

   I.e., check if payoff from immediate exercise > calculated value at that node for corresponding European option

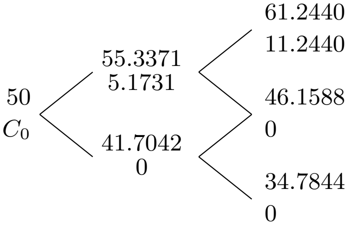
2. If early exercise is optimal, then the payoff from early exercise becomes the new value at that node

3. Continue working backwards through the tree using this procedure

Pricing American Options Example

Example

What would be the price of the call in the previous example if it was an American option?



- Value at node u for European call is 5.1731
- Value at node u from immediate exercise is:

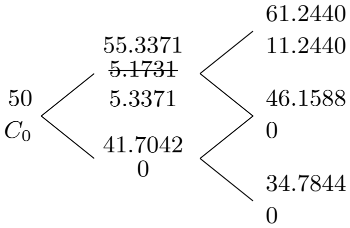
  $$55.3371 - 50 = 5.3371 > 5.1731 \rightarrow \textbf{exercise early}$$

- Call is out of the money at node d, so early exercise not optimal

Pricing American Options Example

Example *(continued)*

- Replace the value of C_u with the payoff from early exercise



- Solve for C_0:

$$C_0 = e^{-.02(.5)}[.4647(5.3371) + 0] = 2.4555$$

- Option is not in the money at node 0, so early exercise is not optimal. The price of our American call is \$2.4555.