**5.** (*7 points*) Let $W_t$ be a standard Weiner process, and $T$ be a fixed time in the future. Define a partition $(t_0, t_1, ..., t_n)$ of the interval $[0, T]$ such that $t_i = \frac{i}{n}T$ for $i = 0, 1, ..., n$.

You are given the following:

I. $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$

II. $E\left(\Delta W_{t_i} e^{\Delta W_{t_i}}\right) = h\, e^{\frac{h}{2}}$ for any $i = 0, 1, 2, ..., n-1$, where $h = \frac{T}{n}$

III. $E\left[\left(e^{W_T - \frac{T}{2}} - 1\right)^2\right] = e^T - 1$

IV. Expected value of a lognormal distribution is $E\left(e^X\right) = e^{\mu + \frac{\sigma^2}{2}}$ where $X \sim N(\mu, \sigma^2)$

Note that

$$E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}\right)^2\right]$$

$$= E\left(\sum_{i=0}^{n-1} e^{2W_{t_i} - t_i} \left(\Delta W_{t_i}\right)^2\right) + 2E\left(\sum_{i<j} e^{W_{t_i} - \frac{t_i}{2} + W_{t_j} - \frac{t_j}{2}} \Delta W_{t_i} \Delta W_{t_j}\right)$$

(a) (*1.5 points*) Calculate $E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}\right)^2\right]$.

(b) (*2.5 points*) Calculate $E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}\right)\left(e^{W_T - \frac{T}{2}} - 1\right)\right]$.

**5.** **Continued**

(c) (*2 points*) Show that $\int_0^T e^{W_s - \frac{s}{2}} dW_S = e^{W_T - \frac{T}{2}} - 1$ by proving that $\sum_{i=0}^{n-1} e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}$ converges to $e^{W_T - \frac{T}{2}} - 1$ in mean square convergence.

(d) (*1 point*) Show that $\int_0^T e^{W_s - \frac{s}{2}} dW_S = e^{W_T - \frac{T}{2}} - 1$ by proving that $d\left( e^{W_t - \frac{t}{2}} \right) = e^{W_t - \frac{t}{2}} dW_t$ using Ito's Lemma.