5. **Learning Objectives:**

1. The candidate will understand the fundamentals of stochastic calculus as they apply to option pricing.

Learning Outcomes:

(1c) Understand Ito integral and stochastic differential equations.

(1d) Understand and apply Ito's Lemma.

Sources:

Nefci Ch. 9

Commentary on Question:

This question tests the candidates' understanding of the fundamentals of stochastic calculus, such as Ito integrals and mean square convergence. The key to solving this question is to understand and apply the properties of Wiener process, specifically the independence of increments and their means and variances, and knowing the definition of mean square convergence. The majority of the candidates struggle with parts (a) to (c), and part (d) is done relatively well.

Solution:

(a) Calculate $E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}} \Delta W_{t_i}\right)^2\right]$.

Commentary on Question:

Most Candidates struggled with parts (a) and (b), as they did not recognize the independence structures within the expressions that they were trying to simplify. Partial marks are given for steps done towards the final answer.

5. **Continued**

$$E\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)^2$$
$$= E\left(\sum_{i=0}^{n-1} e^{2W_{t_i}-t_i}\,(\Delta W_{t_i})^2\right) + 2E\left(\sum_{i<j} e^{W_{t_i}-\frac{t_i}{2}+W_{t_j}-\frac{t_j}{2}}\Delta W_{t_i}\Delta W_{t_j}\right)$$

All terms in the second summation are 0 since ΔW_{t_j} is independent from the rest and has a mean of 0.

Therefore,

$$E\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)^2 = E\left(\sum_{i=0}^{n-1} e^{2W_{t_i}-t_i}\,(\Delta W_{t_i})^2\right) = \sum_{i=0}^{n-1} E\left(e^{2W_{t_i}-t_i}\right)E\left((\Delta W_{t_i})^2\right)$$

Since $W_{t_i}\sim N(0,t_i)$, thus $2W_{t_i}-t_i\sim N(-t_i,4t_i)$, and so

$E\left(e^{2W_{t_i}-t_i}\right)E\left(\Delta W_{t_i}{}^2\right) = e^{-t_i+\frac{4t_i}{2}}h = he^{t_i}$

Or

$E\left(e^{2W_{t_i}-t_i}\right)E\left(\Delta W_{t_i}{}^2\right) = e^{-t_i}E\left(e^{2W_{t_i}}\right)E\left(\Delta W_{t_i}{}^2\right) = e^{-t_i}e^{\frac{4t_i}{2}}h = he^{t_i}$

Therefore,

$$E\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)^2 = \sum_{i=0}^{n-1} he^{t_i} = h\frac{1-e^{nh}}{1-e^h} = (1-e^T)\frac{h}{1-e^h}$$

(b) Calculate $E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)\left(e^{W_T-\frac{T}{2}}-1\right)\right]$.

$$E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)\left(e^{W_T-\frac{T}{2}}-1\right)\right] = \sum_{i=0}^{n-1} E\left[e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\left(e^{W_T-\frac{T}{2}}-1\right)\right]$$

For each term in the summation, denote $A = W_{t_i}$, $B = \Delta W_{t_i}$, $C = W_T - W_{t_{i+1}}$, then

$$E\left[e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\left(e^{W_T-\frac{T}{2}}-1\right)\right] = E\left[e^{A-\frac{t_i}{2}}B\left(e^{(A+B+C)-\frac{T}{2}}-1\right)\right]$$
$$= E\left[e^{2A+B+C-\frac{t_i}{2}-\frac{T}{2}}B\right] - E\left[e^{A-\frac{t_i}{2}}B\right]$$

5. Continued

Due to independence of A, B, and C

$$E\left[e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\left(e^{W_T-\frac{T}{2}}-1\right)\right] = e^{-\frac{t_i}{2}-\frac{T}{2}}E(e^{2A})E(e^C)E(Be^B)$$
$$= e^{-\frac{t_i}{2}-\frac{T}{2}}e^{2t_i}e^{\frac{T-t_{i+1}}{2}}he^{\frac{h}{2}} = he^{t_i}$$

Therefore,

$$E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)\left(e^{W_T-\frac{T}{2}}-1\right)\right] = \sum_{i=0}^{n-1} he^{t_i} = h\frac{1-e^{nh}}{1-e^h} = (1-e^T)\frac{h}{1-e^h}$$

(c) Show that $\int_0^T e^{W_s-\frac{s}{2}}dW_S = e^{W_T-\frac{T}{2}}-1$ by proving that $\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}$ converges to $e^{W_T-\frac{T}{2}}-1$ in mean square convergence.

Commentary on Question:
Partial marks are given for understanding the definition of mean square convergence.

To show that the Ito sum converges in mean square convergence, we need to show:

$$\lim_{n\to\infty} E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)-\left(e^{W_T-\frac{T}{2}}-1\right)\right]^2 = 0$$

$$E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)-\left(e^{W_T-\frac{T}{2}}-1\right)\right]^2$$
$$= E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)^2\right]$$
$$-2E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right)\left(e^{W_T-\frac{T}{2}}-1\right)\right] + E\left[\left(e^{W_T-\frac{T}{2}}-1\right)^2\right]$$
$$= (1-e^T)\frac{h}{1-e^h} - 2(1-e^T)\frac{h}{1-e^h} + e^T - 1$$
$$= -(1-e^T)\frac{h}{1-e^h} + e^T - 1$$

5. Continued

Since $n \to \infty$ is equivalent to $h \to 0$, we calculate the limit of the right hand side as $h \to 0$.
To calculate $\lim_{h\to 0} \frac{h}{1-e^h}$ we need to employ L'Hospital's Rule; using it we see the limit is 1.

$$\lim_{n\to\infty} E\left[\left(\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}}\Delta W_{t_i}\right) - \left(e^{W_T-\frac{T}{2}} - 1\right)\right]^2 = -(1-e^T) + (e^T - 1) = 0$$

(d) Show that $\int_0^T e^{W_s-\frac{s}{2}} dW_S = e^{W_T-\frac{T}{2}} - 1$ by proving that $d\left(e^{W_t-\frac{t}{2}}\right) = e^{W_t-\frac{t}{2}} dW_t$ using Ito's Lemma.

Commentary on Question:
This part is done relatively well, as most candidates were able to apply Ito's Lemma properly and demonstrate how $d\left(e^{W_t-\frac{t}{2}}\right) = e^{W_t-\frac{t}{2}} dW_t$ *leads to proving* $\int_0^T e^{W_s-\frac{s}{2}} dW_S = e^{W_T-\frac{T}{2}} - 1$.

By Ito's Lemma,

$$d\left(e^{W_t-\frac{t}{2}}\right) = \frac{\partial\left(e^{W_t-\frac{t}{2}}\right)}{\partial t}dt + \frac{\partial\left(e^{W_t-\frac{t}{2}}\right)}{\partial W_t}dW_t + \frac{1}{2}\frac{\partial^2\left(e^{W_t-\frac{t}{2}}\right)}{\partial W_t^2}dt$$
$$= -\frac{1}{2}e^{W_t-\frac{t}{2}}dt + e^{W_t-\frac{t}{2}}dW_t + \frac{1}{2}e^{W_t-\frac{t}{2}}dt$$
$$= e^{W_t-\frac{t}{2}}dW_t$$

Therefore, integrating both sides

$$\int_0^T d(e^{W_t-\frac{t}{2}}) = \int_0^T e^{W_t-\frac{t}{2}}dW_t$$
$$e^{W_T-\frac{T}{2}} - e^{0-\frac{0}{2}} = \int_0^T e^{W_t-\frac{t}{2}}dW_t$$
$$\int_0^T e^{W_t-\frac{t}{2}}dW_t = e^{W_T-\frac{T}{2}} - 1$$