5 (a) • First, I use the "Note that" equation given in the question

• The second ~~term~~ will cancel out since ΔW_{t_i} and ΔW_{t_j} are independent and have expectation of 0

Thus, we need to calculate

$$\mathbb{E}\left(\sum_{i=0}^{n-1} e^{2W_{t_i}-t_i} \cdot (\Delta W_{t_i})^2\right)$$

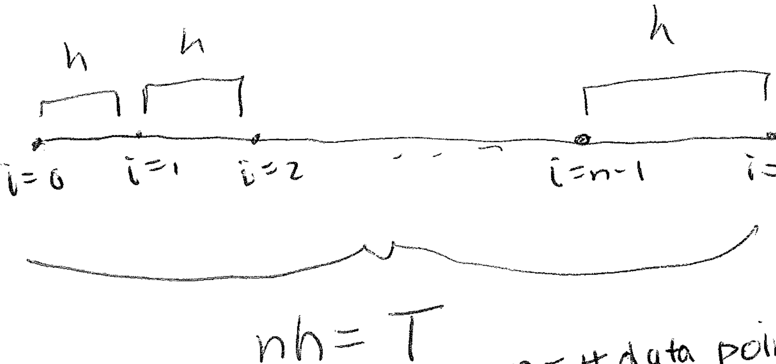
$$= \sum_{i=0}^{n-1} e^{-t_i} \cdot \underbrace{\mathbb{E}\left(e^{2W_{t_i}}\right)}_{\overset{*}{=} e^{2t_i}} \cdot \underbrace{\mathbb{E}\left((\Delta W_{t_i})^2\right)}_{= V(\Delta t_i) = t_{i+1} - t_i = h}$$

$$= \sum_{i=0}^{n-1} h e^{t_i}$$

$$= h \cdot \left(1 + e^{h} + e^{2h} + \dots + e^{(n-1)h}\right)$$

$$= h \cdot \frac{1-e^{nh}}{1-e^{h}}$$

$$\boxed{= h \cdot \frac{1-e^{T}}{1-e^{h}}}$$



n = # data points

h = time between consecutive data points

T = total time

* This comes from the identity

$$\mathbb{E}(e^{kW_t}) = e^{\frac{1}{2}k^2 t}$$ with $k=2$

(b) • The key here is to translate to independent quantities

$W_T - W_{t_{i+1}}$; $W_{t_{i+1}} - W_{t_i}$ and W_{t_i} are all independent

$$\mathbb{E}\left(\left(\sum e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}\right) \cdot \left(e^{W_T - \frac{T}{2}} - 1\right)\right)$$

can cancel -1 in expectation

$$= \mathbb{E}\left(\left(\sum e^{W_{t_i} - \frac{t_i}{2}} \Delta W_{t_i}\right) \cdot \left(e^{W_T - \frac{T}{2}}\right)\right)$$

$$\overset{(*)}{=} \mathbb{E}\left(\sum_{i=0}^{n-1} e^{-\frac{t_i}{2}} e^{-\frac{T}{2}} \underbrace{\Delta W_{t_i} e^{\Delta W_{t_i}}}_{\text{(GIVEN)}} e^{W_T - W_{t_{i+1}}} e^{2W_{t_i}}\right)$$

$$= \sum_{i=0}^{n-1} e^{-\frac{t_i}{2}} e^{-\frac{T}{2}} h e^{h/2} \cdot e^{\frac{1}{2}(T - t_{i+1})} e^{2t_i}$$

$\to$ using $E(e^{kW_t}) = e^{\frac{1}{2}k^2 t}$

$$= \sum_{i=0}^{n-1} h e^{h/2} e^{-\frac{t_i}{2}} e^{-\frac{1}{2}t_{i+1}} e^{2t_i}$$

$$= \sum_{i=0}^{n-1} h e^{h/2} e^{-\frac{1}{2}\underbrace{(t_{i+1} - t_i)}_{h}} e^{t_i}$$

$$= \sum_{i=0}^{n-1} h e^{t_i}$$

same logic as (a)

$$\boxed{= h \cdot \frac{1 - e^T}{1 - e^h}}$$

(*) Using $W_T + W_{t_i}$

$$= W_{t_{i+1}} - W_{t_i} + W_T - W_{t_{i+1}} + 2W_{t_i}$$

$$= \Delta W_{t_i} + W_T - W_{t_{i+1}} + 2W_{t_i}$$

(c) mean square convergence means that : $\lim_{n\to\infty} \underbrace{E\left[f(n)^2\right]}_{g(n)} = 0$

$$\lim_{n\to\infty} E\Big(\underbrace{\sum_{i=0}^{n-1} e^{W_{t_i}-\frac{t_i}{2}} \Delta W_{t_i}}_{A} - \underbrace{\left(e^{W_T-\frac{T}{2}}-1\right)}_{B}\Big)^2 = 0$$

$$E\left[(A-B)^2\right]$$
$$= E(A^2) + E(B^2) - 2E(AB) \qquad (*)$$

$E(A^2) = h \cdot \frac{1-e^T}{1-e^h}$ from (a)

$E(B^2) = e^T - 1$ given (III)

$E(2AB) = 2h \frac{1-e^T}{1-e^h}$ from (b)

Plugging into (*) gives

$$g(n) = -h \cdot \frac{1-e^T}{1-e^h} + e^T - 1 = \frac{h}{1-e^h}\left(e^T-1\right) + e^T - 1$$

as $n\to\infty$, $h\to 0$

$$\lim_{h\to 0} \frac{h}{1-e^h} = \lim_{h\to 0} \frac{1}{-e^h} = \frac{1}{-1} = -1$$

Thus,

$$\lim_{n\to\infty} g(n) = (-1)(e^T-1) + e^T - 1 = \underline{\underline{0}} \qquad \blacksquare$$

(d) Simple application of Baby Itô's Lemma

Step 1 $\rightarrow$ Baby Itô

Step 2 $\rightarrow$ Integrate Step (1)

(1) Let $F = e^{W_t - \frac{t}{2}}$ $\quad \partial F/\partial t = -\frac{1}{2}F$

$\partial F/\partial W_t = \partial^2 F/\partial W_t^2 = F$

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W_t^2}\right) dt + \frac{\partial F}{\partial W_t} dW_t$$

$$= \left(-\frac{1}{2}F + \frac{1}{2}F\right) dt + F dW_t$$

$\Rightarrow dF = F dW_t$

$\Rightarrow d\left(e^{W_t - \frac{t}{2}}\right) = e^{W_t - \frac{t}{2}} dW_t$

(2) Integrating gives

$$e^{W_t - \frac{t}{2}} = \int_0^t e^{W_s - \frac{s}{2}} dW_s$$

Plugging in $t = T$ gives the desired result