Binomial Interest Rates - Outline

Bond Forward Prices

Let $P(T, T+s)$ be the time T price of a \$1 zero-coupon bond maturing at time $T+s$. Then the **forward price** at time t of a bond maturing at time $T+s$ and delivered at time T is given by:

$$F_{t,T}[P(T, T+s)] = \frac{P(t, T+s)}{P(t, T)}$$

The price of a *prepaid* forward on the bond is of course:

$$F^P_{t,T}[P(T, T+s)] = P(t, T+s)$$

Let $R_t(T, T+s)$ be the **forward rate**, i.e. the non-annualized effective interest rate on a loan one can obtain at time t for a risk-free loan that starts at time T and is repaid at time $T+s$. The forward rate must solve:

$$\frac{1}{1+R_t(T, T+s)} = F_{t,T}[P(T, T+s)]$$

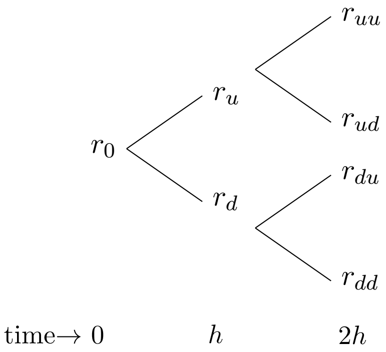
$$\Updownarrow$$

$$R_t(T, T+s) = \frac{1}{F_{t,T}[P(T, T+s)]} - 1$$

If $s = 1$, then the formula gives the annual forward rate. Otherwise, you must annualize it if required.

Binomial Interest Rate Trees

Consider a binomial tree for continuously compounded interest rates, where one binomial step is of length h



Two notes:

1. Interest rate trees sometimes don't recombine
2. Rates in the tree are "forward-looking." I.e., r_0 is the relevant interest rate for the period 0 to h

Binomial Bond Pricing

Because rates in the tree look forward, to price a bond maturing at time T, we need only create a tree out to time $T - h$

The price of a \$1 bond maturing at time T at node j at time $T - h$ is:

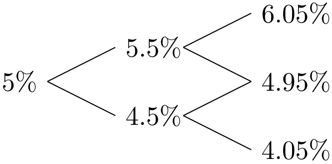
$$P_j(T - h, T) = e^{-r_j h}$$

Once the bond prices at each node at time $T - h$ are known, the price of the bond today is the risk-neutral expected present value of the time $T - h$ bond price

- Note that because interest rates change throughout the tree, the present value of the time $T - h$ bond price in a recombining tree will differ for each unique path to node j
- Thus, the present value of each time $T - h$ price must be calculated separately for each unique path to that node

Example

Consider the following tree for continuously compounded interest rates. Let $h = 1$ year and the risk-neutral probability of an up move be 75%. Find $P(0, 3)$.



1. Find bond prices at time $T - h$

$$P_{uu}(2,3) = e^{-.0605}$$
$$P_{ud}(2,3) = P_{du}(2,3) = e^{-.0495}$$
$$P_{dd}(2,3) = e^{-.0405}$$

Example *(continued)*

2. Calculate the risk-neutral expected present value of the time $T - h$ bond price

$$\begin{aligned} P(0,3) &= \big((.75)^2 e^{-.0605} e^{-.055} e^{-.05} \\ &\quad + .75(.25) e^{-.0495} e^{-0.055} e^{-.05} \\ &\quad + .25(.75) e^{-.0495} e^{-.045} e^{-.05} \\ &\quad + (.25)^2 e^{-.0405} e^{-.045} e^{-.05}\big) \\ &= 0.8542 \end{aligned}$$

To speed up calculation, note that the price gets discounted by r_0 along all paths

A bond's **yield-to-maturity** (often simply called its yield) is the constant per-period discount rate that results in the bond's price

E.g., the yield of the 3-year bond in our previous example, $y(0,3)$, solves:

$$0.8542 = e^{-3\,y(0,3)}$$

$$y(0,3) = \frac{\ln(.8542)}{-3} = .0525$$

Note that a bond's yield $= r_0$ if either of the following is true:

- A bond matures in one binomial period, or
- Interest rates are constant