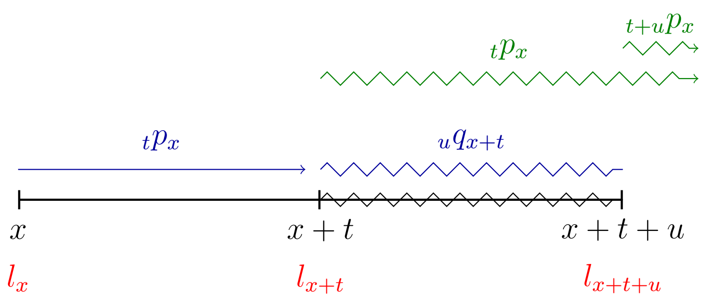
Last updated April 1, 2019.

1. [First Pass] Which of the following are true?

   1. ${}_{t|u}q_x = {}_tp_x \cdot {}_uq_{x+t}$
   2. ${}_{t|u}q_x = \frac{l_{x+t+u} - l_{x+t}}{l_x}$
   3. ${}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x$

A. 1 B. 2 C. 3 D. $1, 2$ E. $1, 3$

Draw a picture for ${}_{t|u}q_x$



(i) ${}_{t|u}q_x = {}_tp_x \cdot {}_uq_{x+t}$ ✓

(ii) ${}_{t|u}q_x = \frac{l_{x+t} - l_{x+t+u}}{l_x} \neq \frac{l_{x+t+u} - l_{x+t}}{l_x}$

(iii) ${}_{t|u}q_x = {}_tp_x - {}_{t+u}p_x$ ✓

Thus, $\boxed{1, 3}$ are true.

2. [First Pass] Given that a life aged 50 will live to age 60, what is the probability p that he will die between ages 70 and 80?

Age	l_x
50	89,509
60	81,881
70	66,162
80	39,144

A. Less than 0.310
B. At least 0.310, but less than 0.315
C. At least 0.315, but less than 0.320
D. At least 0.320, but less than 0.325
E. At least 0.325

$$p = \frac{l_{70} - l_{80}}{l_{60}} = \frac{66{,}162 - 39{,}144}{81{,}881} = \boxed{0.33}$$

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3. [First Pass] You are given the following mortality table:

Age(x)	q_x	l_x	d_x
20		30,000	1,200
21			
22		27,350	
23	0.0700		
24	0.0790	23,900	

Determine the probability that a life aged 21 will die within two years.

A. Less than 0.0960
B. At least 0.0960, but less than 0.1010
C. At least 0.1010, but less than 0.1060
D. At least 0.1060, but less than 0.1110
E. At least 0.1110

$$ {}_2q_{21} = \frac{l_{21} - l_{23}}{l_{21}} $$
$$ l_{21} = l_{20} - d_{20} = 30000 - 1200 = 28800 $$

$$l_{23} \times p_{23} = l_{24}$$
$$l_{23} = \frac{23900}{0.93} = 25699$$
$$_2q_{21} = \frac{28800 - 25699}{28800} = \boxed{0.1077}$$

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4. [First Pass] Given the following portion of a life table:

x	l_x	d_x	p_x	q_x
0	1,000		0.875	
1				
2	750			0.25
3				
4				
5	200	120		
6				
7		20		1.00

Determine the value of $p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6$.

A. Less than 0.055
B. At least 0.055, but less than 0.065
C. At least 0.065, but less than 0.075
D. At least 0.075
E. The answer cannot be determined from the given information.

First simplify the expression

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6 = {}_5p_1 \cdot q_6 = {}_{5|1}q_1 = \frac{l_6 - l_7}{l_1}$$

Now find l_1, l_6 and l_7

$$l_1 = 1000(0.875) = 875$$
$$l_6 = 200 - 120 = 80$$
$$l_7 = 20$$

Plugging these back into the first equation we have

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6 = \frac{80 - 20}{875} = \boxed{0.06857}$$

5. [First Pass] You are given $S_0(x) = \dfrac{1}{1+x}$.

Determine the median future lifetime of (y).

A. $y+1$ B. y C. 1 D. $\frac{1}{y}$ E. $\frac{1}{1+y}$

Let m be the median

$$
\begin{aligned}
{}_m p_y &= \frac{S_0(y+m)}{S_0(y)} \\
0.5 &= \frac{\frac{1}{1+y+m}}{\frac{1}{1+y}} \\
0.5 &= \frac{1+y}{1+y+m} \\
2(1+y) &= 1+y+m \\
\boxed{1+y} &= m
\end{aligned}
$$

6. You are given the following information:

(i) $l_1 = 9700$

(ii) $q_1 = q_2 = 0.020$

(iii) $q_4 = 0.026$

(iv) $d_3 = 232$

Determine the expected number of survivors to age 5.

A. Less than 8,845
B. At least 8,845, but less than 8,850
C. At least 8,850, but less than 8,855
D. At least 8,855, but less than 8,860
E. At least 8,860

$$\begin{aligned} l_2 &= l_1\, p_1 = 9700(1 - .02) = 9506 \\ l_3 &= l_2\, p_2 = 9506(1 - .02) = 9315.88 \\ l_4 &= l_3 - d_3 = 9315.88 - 232 = 9083.88 \\ l_5 &= l_4\, p_4 = 9083.88(1 - .026) = \boxed{8847.70} \end{aligned}$$

7. You are given the following information:

(i) The probability that two 70-year-olds are both alive in 20 years is 16%.

(ii) The probability that two 80-year-olds are both alive in 20 years is 1%.

(iii) There is an 8% chance of a 70-year-old living 30 years.

(iv) All lives are independent and have the same expected mortality.

Determine the probability of an 80-year-old living 10 years.

A. Less than 0.35
B. At least 0.35, but less than 0.45
C. At least 0.45, but less than 0.55
D. At least 0.55, but less than 0.65
E. At least 0.65

Set $l_{70} = 100$ (arbitrary). We are given:

(i) $l_{90} = l_{70}\sqrt{0.16} = 100(0.4) = 40$

(ii) $l_{100} = l_{80}\sqrt{0.01} = 0.1\, l_{80}$

(iii) $l_{100} = l_{70}(0.08) = 100(0.08) = 8$

Combining 2 and 3 we have

$$\begin{aligned} 0.1\, l_{80} &= 8 \\ l_{80} &= 80 \end{aligned}$$

Finally

$$ {}_{10}p_{80} = \frac{l_{90}}{l_{80}} = \frac{40}{80} = \boxed{0.5}$$

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8. Light bulbs burn out according to the following life table:

x	l_x
0	1,000,000
1	800,000
2	600,000
3	300,000
4	0

A new plant has 2,500 light bulbs. Burned out light bulbs are replaced with new light bulbs at the end of each year.

What is the expected number of new light bulbs that will be needed at the end of year 3?

A. Less than 800
B. At least 800, but less than 860
C. At least 860, but less than 920
D. At least 920, but less than 980
E. At least 980

Completing the table we have

x	l_x	d_x	$_{x\vert}q_0$
0	1,000,000	200,000	$\frac{200{,}000}{1{,}000{,}000} = 0.2$
1	800,000	200,000	$\frac{200{,}000}{1{,}000{,}000} = 0.2$
2	600,000	300,000	$\frac{300{,}000}{1{,}000{,}000} = 0.3$
3	300,000	300,000	$\frac{300{,}000}{1{,}000{,}000} = 0.3$

So of the original 2,500 bulbs 20% burn out during first year, 20% during second year and 30% during the third year.

Year 1	Year 2	Year 3
$2{,}500(0.2) = 500$	$2{,}500(0.2) = 500$	$2{,}500(0.3) = 750$

Now we need to account for the 500 bulbs that burned out during the first year. 20% of these burn out during year 2 (one year later) and 20% burn out during year 3 (two years later)

Year 1	Year 2	Year 3
2,500(0.2) = 500	2,500(0.2) = 500 500(0.2) = 100	2,500(0.3) = 750 500(0.2) = 100

Now we account for the $500 + 100 = 600$ bulbs that burned out during year 2. 20% of those will burn out during the third year (one year later). So our final table looks like

Year 1	Year 2	Year 3
2,500(0.2) = 500	2,500(0.2) = 500 500(0.2) = 100	2,500(0.3) = 750 500(0.2) = 100 600(0.2) = 120

So the total number of bulbs replaced during the 3rd year $= 750 + 100 + 120 = 970$.

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9. [SOA.MLC.200] The graph of a piecewise linear survival function, $S_0(t)$, consists of 3 line segments with endpoints $(0, 1)$, $(25, 0.50)$, $(75, 0.40)$, $(100, 0)$.

   Calculate $\dfrac{{}_{20|55}q_{15}}{{}_{55}q_{35}}$.

   A. 0.69 B. 0.71 C. 0.73 D. 0.75 E. 0.77

Build a life table

x	ℓ_x
0	100 (arbitrary)
25	$100(0.5) = 50$
75	$100(0.4) = 40$
100	0

$$\frac{{}_{20|55}q_{15}}{{}_{55}q_{35}} = \frac{\frac{\ell_{35}-\ell_{90}}{\ell_{15}}}{\frac{\ell_{35}-\ell_{90}}{\ell_{35}}} = \frac{\ell_{35}}{\ell_{15}} = \frac{48}{70} = \boxed{0.6857}$$

where ℓ_{35} and ℓ_{70} are found using linear interpolation (because we given a piecewise linear survival function)

$$\ell_{15} = \frac{10}{25}(100) + \frac{15}{25}(50) = 70$$

$$\ell_{35} = \frac{40}{50}(50) + \frac{10}{50}(40) = 48$$

10. [3.S00.28] For a mortality study on college students:

(i) Students entered the study on their birthdays in 1963.

(ii) You have no information about mortality before birthdays in 1963.

(iii) Dick, who turned 20 in 1963, died between his 32nd and 33rd birthdays.

(iv) Jane, who turned 21 in 1963, was alive on her birthday in 1998, at which time she left the study.

(v) All lifetimes are independent.

(vi) Likelihoods are based upon the Standard Ultimate Life Table.

Calculate the likelihood for these two students.

A. 0.00029 B. 0.00033 C. 0.00039 D. 0.00043 E. 0.00049

$$ {}_{12|}q_{20} \cdot {}_{35}p_{21} = \left(\frac{l_{32} - l_{33}}{l_{20}}\right)\left(\frac{l_{56}}{l_{21}}\right) = \boxed{0.000331} $$

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11. [SOA 3.4; adapted from MLC.F13.24] The SULT Club has 4000 members all age 25 with independent future lifetimes. The mortality for each member follows the Standard Ultimate Life Table.

Calculate the largest N, using the normal approximation, such that the probability that there are at least N survivors at age 95 is at least 90%.

A. 800 B. 815 C. 830 D. 845 E. 860

Let X be the number of survivors at age 95, then $X \sim \text{Binomial}(n = 4000, p = {}_{70}p_{25})$. Using SULT, we can determine

$$ {}_{70}p_{25} = \frac{l_{95}}{l_{25}} = \frac{21{,}178.3}{99{,}871.1} = 0.2121 $$

Hence, mean and variance is given by

$$\mu = np = (4{,}000)(0.2121) = 848.4$$
$$\sigma^2 = np(1-p) = (4{,}000)(0.2121)(1-0.2121) = 668.4544$$

Therefore, to find the largest N such that $\mathrm{P}(X \geq N) \geq 0.9$ is equivalent to solve for

$$\mathrm{P}(X \geq N) = \mathrm{P}\left(\frac{X-\mu}{\sigma} \geq \frac{N-\mu}{\sigma}\right) = \mathrm{P}\left(Z \geq \frac{N-(848.4)}{\sqrt{668.4544}}\right) = 0.9$$

Using the normal table to find the critical value z such that $\mathrm{P}(Z < z) = 0.1$, which results in

$$\frac{N-(848.4)}{25.8545} = -1.282 \implies N = \boxed{815.2546}$$

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12. [SOA 2.2; MLC.S13.20] Scientists are searching for a vaccine for a disease. You are given:

(i) 100,000 lives age x are exposed to the disease.

(ii) Future lifetimes are independent, except that the vaccine, if available, will be given to all at the end of year 1.

(iii) The probability that the vaccine will be available is 0.2.

(iv) For each life during year 1, $q_x = 0.02$.

(v) For each life during year 2, $q_{x+1} = 0.01$ if the vaccine has been given, and $q_{x+1} = 0.02$ if it has not been given.

Calculate the standard deviation of the number of survivors at the end of year 2.

A. 100 B. 200 C. 300 D. 400 E. 500

Since we have two sources of randomness, the individual mortalities and the vaccine availability, conditional variation can be used.

If the vaccine is available, then the expected number of survivors at the end of year 2 is $100000(1-0.02)(1-0.01) = 97020$. Since each person in this case survives independently with probability $0.98 \cdot 0.99$ and dies otherwise, we can use the Bernoulli shortcut on individuals and the fact that the variance of the sum for independent variables is the sum of the variances to find that the variance of the number of survivors when the vaccine is available is $100000(0.98 \cdot 0.99)(1-0.98 \cdot 0.99) = 2891.196$.

If the vaccine is not available, analogously, the expected number of survivors is $100000(1-0.02)(1-0.02) = 96040$, and the variance is $100000(0.98 \cdot 0.98)(1-0.98 \cdot 0.98) = 3803.184$.

Now we can use the conditional variance. I'll call the number of survivors S, and the vaccine variable V. Since there are only two values for the expected value given V, we can use the Bernoulli shortcut again to find this variance.

$$\begin{aligned}\text{Var}(S) &= E_V\left[\text{Var}(S \mid V)\right] + \text{Var}_V\left(E[S \mid V]\right) \\ &= (2891.196 \cdot 0.2 + 3803.184(1-0.2)) + (97020 - 96040)^2 \cdot 0.2(1-0.2) = 157284.7864.\end{aligned}$$

The standard deviation is the square root of this variance, $\boxed{396.591}$, closest to answer choice D.

The video solution also demonstrates how to compute the variance of the expected value and the expected value of the variances using the calculator data function.

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13. [SOA 3.10; MLC.F15.02] A group of 100 people start a Scissor Usage Support Group. The rate at which members enter and leave the group is dependent on whether they are right-handed or left-handed.

    You are given the following:

    (i) The initial membership is made up of 75% left-handed members (L) and 25% right-handed members (R).

    (ii) After the group initially forms, 35 new (L) and 15 new (R) join the group at the start of each subsequent year.

    (iii) Members leave the group only at the end of each year.

    (iv) $q^L = 0.25$ for all years.

    (v) $q^R = 0.50$ for all years.

    Calculate the proportion of the Scissor Usage Support Group's expected membership that is left-handed at the start of the group's 6^{th} year, before any new members join for that year.

    A. 0.76 B. 0.81 C. 0.86 D. 0.91 E. 0.96

Note that the initial makeup of the group is not random, but rather exactly 75% left-handed people, for an initial total of 75 of 100 people. Since 25% of these leave in each subsequent year, the number expected to still be around after 5 years is $75(1-0.25)^5 \approx 17.19785$. The 35 new lefties that arrive at time 1 only need to survive for 4 years, and so on, so that we can find the number of lefties at time 5 by

$$75(0.75)^5 + 35(0.75)^4 + 35(0.75)^3 + 35(0.75)^2 + 35(0.75)^1 \approx 89.58.$$

The computation for the right-handed folks is similar, but we start with only 25 and add only 15 new members each year:

$$25(0.5)^5 + 15(0.5)^4 + 15(0.5)^3 + 15(0.5)^2 + 15(0.5)^1 \approx 14.84.$$

Finally the proportion of lefties at time 5, just before year 6 new members arrive, is $89.58/(89.58 + 14.84) = \boxed{0.858}$, closest to answer choice C.
