A.5 Frequency Models - Outline

- A.5.2 Poisson-Gamma
  - Connection to Negative Binomial
  - Example and Exercises

Poisson/Gamma Pairs

Suppose that $N \sim \text{Poisson}(\lambda)$ where $\lambda \sim \text{Gamma}(\alpha, \theta)$. Then

$$\begin{aligned} \mathrm{E}[N] &= \mathrm{E}[\mathrm{E}[N \mid \lambda]] = \mathrm{E}[\lambda] = \alpha\theta \\ \mathrm{Var}[N] &= \mathrm{E}[\mathrm{Var}[N \mid \lambda]] + \mathrm{Var}[\mathrm{E}[N \mid \lambda]] \\ &= \mathrm{E}[\lambda] + \mathrm{Var}[\lambda] \\ &= \alpha\theta + \alpha\theta^2 = \alpha\theta(1+\theta) \end{aligned}$$

A negative binomial with parameters $r = \alpha$ and $\beta = \theta$ has the same moments. Hmm.

It turn out that $N \sim \text{Negative Binomial}(r = \alpha, \beta = \theta)$

Key point: The parameter values are the same as for the Gamma.

Proof

$N \sim \text{Poisson}(\lambda)$ where $\lambda \sim \text{Gamma}(\alpha, \theta)$

$$\begin{aligned}
\text{P}[N = k] &= \int_0^\infty \text{P}[N = k \mid \lambda] \cdot f(\lambda) d\lambda \\
&= \int_0^\infty e^{-\lambda} \frac{\lambda^k}{k!} \cdot \frac{\lambda^{\alpha-1}}{\theta^\alpha \Gamma(\alpha)} e^{-\lambda/\theta} d\lambda \\
&= \frac{1}{\theta^\alpha k! \Gamma(\alpha)} \int_0^\infty \lambda^{k+\alpha-1} e^{-\lambda/[\theta/(\theta+1)]} d\lambda \\
&= \frac{1}{\theta^\alpha k! \Gamma(\alpha)} \cdot \frac{\theta}{\theta+1} \int_0^\infty \frac{\lambda^{k+\alpha-1}}{\theta/(\theta+1)} e^{-\lambda/[\theta/(\theta+1)]} d\lambda \\
&= \frac{\Gamma(k+\alpha)}{k! \Gamma(\alpha)} \left(\frac{\theta}{\theta+1}\right)^{\alpha+k} \frac{1}{\theta^\alpha} \\
&= \frac{\alpha(\alpha+1)\dots(\alpha+k-1)}{k!} \cdot \frac{\theta^k}{(1+\theta)^{\alpha+k}}
\end{aligned}$$

Example

The number of claims for a single policyholder follows a Poisson distribution with mean λ, where λ follows a Gamma distribution. The number of claims for a policyholder chosen at random has mean 0.2 and variance 0.3.
Determine the variance of the Gamma distribution.
By the Poisson/Gamma trick, we know that N is a negative binomial.

$$\begin{aligned}
\text{E}[N] &= r\beta = 0.2 \\
\text{Var}[N] &= r\beta(1+\beta) = 0.3 \\
1 + \beta &= \frac{0.3}{0.2} = 1.5 \\
\beta = 0.5 \qquad r &= 0.4
\end{aligned}$$

$$\lambda \sim \text{Gamma}(\alpha = 0.4, \theta = 0.5)$$

$$\text{Var}(\lambda) = \alpha\theta^2 = \boxed{0.1}$$

Exercise 1

The distribution of the number of claims, N, given Λ, is a Poisson with mean Λ. The distribution of Λ is exponential with mean 2. Find $\mathrm{P}[N = 0]$.

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$$\Lambda \sim \text{Gamma}(\alpha = 1, \theta = 2)$$

$$N \sim \text{Neg. Bin.}(r = 1, \beta = 2)$$

$$\begin{aligned} \mathrm{P}[N = 0] &= \frac{1}{(1+\beta)^r} \\ &= \boxed{\frac{1}{3}} \end{aligned}$$

Exercise 2

You are given:

1. The claim count N has a Poisson distribution with mean L.
2. L has a Gamma distribution with mean 1 and variance 2.

Find the variance of N.

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$$
\begin{aligned}
\mathrm{E}[L] &= \alpha\theta = 1 \\
\mathrm{Var}[L] &= \alpha\theta^2 = 2 \\
\frac{\mathrm{Var}[L]}{\mathrm{E}[L]} &= \theta = 2 \quad \Rightarrow \quad \alpha = \frac{1}{2} \\
r = \alpha = \frac{1}{2} &\quad \beta = \theta = 2 \\
\mathrm{Var}[N] &= r\beta(1+\beta) \\
&= \frac{1}{2} \cdot 2 \cdot 3 = \boxed{3}
\end{aligned}
$$