The Infinite Actuary Exam STAM Online Course
A.5.2. Poisson-Gamma Mixtures
Last updated April 11, 2018

1. An actuary for an automobile insurance company determines that the distribution of the annual number of claims for an insured chosen at random is modeled by the negative binomial distribution with mean 0.2 and variance 0.3.

   The number of claims for each individual insured has a Poisson distribution and the means of these Poisson distributions are gamma distributed over the population of insureds.

   Calculate the variance of this gamma distribution.

   A. 0.10 B. 0.15 C. 0.20 D. 0.25 E. 0.30

Let N be the number of claims. $\text{E}[N] = r\beta = 0.2$, and $\text{Var}[N] = r\beta(1+\beta) = 0.3$, so $0.3/0.2 = 1+\beta$ and $\beta = 0.5$, $r = 0.4$.

The parameterization used in the Loss Models text for negative binomial distribution is such that the parameters of the Gamma in a Poisson-Gamma pair are the same as the parameters of the negative binomial, with $\alpha = r$ and $\theta = \beta$. $\text{Var}(\Lambda) = \alpha\theta^2 = 0.4 \cdot 0.5^2 = \boxed{0.1}$

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2. [3-CAS.F04.21] The number of auto claims for a group of 1,000 insured drivers has a negative binomial distribution with $\beta = 0.5$ and $r = 5$.

   Determine the parameters β and r for the distribution of the number of auto claims for a group of 2,500 such individuals.

   A. $\beta = 1.25$ and $r = 5$
   B. $\beta = 0.20$ and $r = 5$
   C. $\beta = 0.50$ and $r = 5$
   D. $\beta = 0.20$ and $r = 12.5$
   E. $\beta = 0.50$ and $r = 12.5$

   If we increase the number of individuals, β remains fixed and we simply update r. $r_{new}/r_{old} = 2500/1000 = 2.5$, so $r_{new} = 2.5 \cdot 5 = 12.5$, giving us answer $\boxed{E}$

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3. [3.F02.5] Actuaries have modeled auto windshield claim frequencies. They have concluded that the number of windshield claims filed per year per driver follows the Poisson distribution with parameter λ, where λ follows the gamma distribution with mean 3 and variance 3. Calculate the probability that a driver selected at random will file no more than 1 windshield claim next year.

   A. 0.15 B. 0.19 C. 0.20 D. 0.24 E. 0.31

The distribution of λ is a Gamma with $\text{E}\lambda = \alpha\theta = 3$ and $\text{Var}\lambda = \alpha\theta^2 = 3$, so $\theta = 1$ and $\alpha = 3$.

N is therefore a negative binomial with $r = \alpha = 3$ and $\beta = \theta = 1$, so
$\mathrm{P}[N \le 1] = \dfrac{1}{(1+1)^3} + \dfrac{3}{1!}\dfrac{1}{(1+1)^4} = \boxed{0.3125}$

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4. [3.S00.4] You are given:

   (i) The claim count N has a Poisson distribution with mean L .

   (ii) L has a gamma distribution with mean 1 and variance 2.

   Calculate the probability that $N = 1$.

   A. 0.19 B. 0.24 C. 0.31 D. 0.34 E. 0.37

$\mathrm{E}[L] = \alpha\theta = 1$ and $\mathrm{Var}[L] = \alpha\theta^2 = 2$, so $\theta = 2$ and $\alpha = 1/2$, so N is a negative binomial with $r = 1/2$ and $\beta = 2$, so $\mathrm{P}[N = 1] = \dfrac{(1/2)2^1}{1!(1+2)^{3/2}} = \dfrac{1}{3^{1/2}} = \boxed{0.1925}$

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5. [3.Sample.12] The annual number of accidents for an individual driver has a Poisson distribution with mean λ. The Poisson means, λ, of a heterogeneous population of drivers has a gamma distribution with mean 0.1 and variance 0.01.

   Calculate the probability that a driver selected at random from the population will have 2 or more accidents in one year.

   A. 1/121 B. 1/110 C. 1/100 D. 1/90 E. 1/81

Since we have a Poisson/Gamma, the number of accidents for a randomly selected driver is negative binomial$(r = \alpha, \beta = \theta)$.

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\begin{aligned}
\alpha\theta &= 0.1 \\
\alpha\theta^2 &= 0.01 \\
\theta &= 0.1 \quad \alpha = 1 \\
N &\sim \text{Neg. Bin.}(r = 1, \beta = 0.1) = \text{Geo.}(\beta = 0.1) \\
\mathrm{P}[N \ge 2] &= 1 - \mathrm{P}[N = 0] - \mathrm{P}[N = 1] \\
&= 1 - \frac{1}{1 + 0.1} - \frac{0.1}{1.1^2} = \boxed{\frac{1}{121}}
\end{aligned}
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6. [4B.F96.26] You are given the following:

- The probability that a single insured will produce 0 claims during the next exposure period is $e^{-\lambda}$
- λ varies by insured and follows a distribution with density function $f(\lambda) = 36\lambda e^{-6\lambda}, 0 < \lambda < \infty$

Determine the probability that a randomly selected insured will produce 0 claims during the next exposure period.

A. Less than 0.72
B. At least 0.72, but less than 0.77
C. At least 0.77, but less than 0.82
D. At least 0.82, but less than 0.87
E. At least 0.87

We only know $\mathrm{P}[N = 0 \mid \lambda] = e^{-\lambda}$, not the entire distribution of N. But we only want $\mathrm{P}[N = 0]$. The mathematically correct approach is

$$\begin{aligned}
\mathrm{P}[N = 0] &= \int_0^\infty \mathrm{P}[N = 0 \mid \lambda] \cdot f(\lambda) d\lambda \\
&= \int_0^\infty e^{-\lambda} \cdot 36\lambda e^{-6\lambda} d\lambda \\
&= 36 \int_0^\infty \lambda e^{-7\lambda} d\lambda \\
&= \frac{36}{7} \int_0^\infty \lambda 7 e^{-7\lambda} d\lambda \\
&= \boxed{\frac{36}{49}}
\end{aligned}$$

where to evaluate the integral, I am rewriting it as something that looks like the mean of an exponential. You can also use the "gamma trick" if you have seen that.

The exam approach is to pretend that $(N \mid \lambda)$ is Poisson since the probabilities match at 0 and we don't care about any other values. In that case, N is negative binomial and

$$\begin{aligned}
\lambda &\sim \text{Gamma}(\alpha = 2, \theta = 1/6) \\
N &\sim \text{Neg. Bin.}(r = 2, \beta = 1/6) \\
\mathrm{P}[N = 0] &= \frac{1}{(1 + 1/6)^2} = \boxed{\frac{36}{49}}
\end{aligned}$$

7. [3.S05.10] Low Risk Insurance Company provides liability coverage to a population of 1,000 private passenger automobile drivers. The number of claims during a given year from this population is Poisson distributed.

If a driver is selected at random from the population, his expected number of claims per year is a random variable with a Gamma distribution such that $\alpha = 2$ and $\theta = 1$.

Calculate the probability that a driver selected at random will not have a claim during the year.

A. 11.1% B. 13.5% C. 25.0% D. 33.3% E. 50.0%

Let λ be the expected number of claims for a random driver, and N the observed number of claims.

$$\lambda \sim \text{Gamma}(\alpha = 2, \theta = 1)$$
$$N \sim \text{Neg. Bin.}(r = 2, \beta = 1)$$
$$\text{P}[N = 0] = \frac{1}{(1+\beta)^r} = \frac{1}{2^2} = \boxed{0.25}$$

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8. [3.S00.4] You are given:

(i) The claim count N has a Poisson distribution with mean Λ.

(ii) Λ has a gamma distribution with mean 1 and variance 2.

Calculate the unconditional probability that $N = 1$.

A. 0.19 B. 0.24 C. 0.31 D. 0.34 E. 0.37

We have a Poisson/Gamma pair, so the unconditional distribution of N is a negative binomial distribution. First, we need to find the parameters.

$$\text{E}[\Lambda] = 1 = \alpha\theta$$
$$\text{Var}[\Lambda] = 2 = \alpha\theta^2 = (\alpha\theta)\theta$$

so $\theta = 2$ and $\alpha = 1/2$. Our negative binomial has the same values of its parameters, so $r = 1/2$ and $\beta = 2$. That gives

$$\text{P}[N = 1] = \frac{r}{1} \cdot \frac{\beta^1}{(1+\beta)^{r+1}} = \frac{(0.5)2}{3^{1.5}} = \boxed{0.192}$$

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9. [3.S01.15] An actuary for an automobile insurance company determines that the distribution of the annual number of claims for an insured chosen at random is modeled by the negative binomial distribution with mean 0.2 and variance 0.4.

The number of claims for each individual insured has a Poisson distribution and the means of these Poisson distributions are gamma distributed over the population of insureds.

Calculate the variance of this gamma distribution.

A. 0.20 B. 0.25 C. 0.30 D. 0.35 E. 0.40

N is a negative binomial with $\mathrm{E}[N] = r\beta = 0.2$ and $\mathrm{Var}[N] = r\beta(1+\beta) = 0.4$, so $1+\beta = 2$ and $\beta = 1, r = 0.2$.

That also means that N is a Poisson with mean Λ, where Λ is a Gamma with $\alpha = 0.2$ and $\theta = 1$, so $\mathrm{Var}(\lambda) = \alpha\theta^2 = (0.2)1^2 = \boxed{0.2}$