C.1 Maximum Likelihood Estimators - Outline

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Example

Ground up losses are exponential(θ), and losses below the deductible are not reported. Find the MLE of θ given the following:

Deductible Type	Ordinary	Ordinary	Franchise	Franchise
Deductible Amount	100	200	150	250
Payment Amount	50	300	200	400
Loss Amount	150	500	200	400

$$L(\theta) = \frac{f(150)}{S(100)} \cdot \frac{f(500)}{S(200)} \cdot \frac{f(200)}{S(150)} \cdot \frac{f(400)}{S(250)}$$

$$= \frac{1}{\theta} e^{-(150-100)/\theta} \cdot \frac{1}{\theta} e^{-(500-200)/\theta} \cdot \ldots$$

$$\ell(\theta) = -4\ln(\theta) - \frac{50+300+50+150}{\theta}$$

$$\ell'(\theta) = 0 = \frac{-4}{\theta} + \frac{50+300+50+150}{\theta^2}$$

$$\widehat{\theta} = \frac{50+300+50+150}{4} = \boxed{137.5}$$

General Formula

Suppose that we have individual data and $X \sim \exp(\theta)$. Then

1. With complete data, $\widehat{\theta} = \overline{X} = \dfrac{\sum \text{losses}}{\# \text{ of losses}}$
2. With incomplete data, then this generalizes to
$$\widehat{\theta} = \frac{\sum \text{observed exposure}}{\# \text{ of observed (i.e., non-censored) losses}}$$

Observed exposure = loss/censoring amount − truncation value

If $X = 500$ and is left-truncated at $d = 100$, then we observe from 100 to 500. Observed exposure $= 500 - 100 = 400$.

If $X = 600^+$ (censored at 600) and truncate at $d = 100$, then we observe from 100 to 600. Observed exposure $= 600 - 100 = 500$. *This data point is not counted in the denominator.*

With grouped data, we have to compute likelihood functions.

Example (Revisited)

You observe the following 4 payments:

Deductible Type	Ordinary	Ordinary	Franchise	Franchise
Deductible Amount	100	200	150	250
Payment Amount	50	300	200	400

Losses below the deductible are not reported. The ground up loss distribution is exponential(θ). What is the MLE of θ?

For an ordinary deductible, payment = observed exposure.
For a franchise deductible, payment = loss
Observed exposure = loss − deductible

Our observed exposures are $50, 300, (200 - 150), (400 - 250)$.
$$\widehat{\theta} = \frac{50 + 300 + 50 + 150}{4} = \boxed{137.5}$$

Example

Losses are exponential with mean θ. 10 losses are observed that exceed a deductible of 2. Of those 10 losses, 4 exceed the maximum covered loss of 7. What is the MLE of θ?

$$\begin{aligned}
L(\theta) &= (\mathrm{P}[2 < X \le 7 \mid X > 2])^6 \cdot (\mathrm{P}[7 < X \mid X > 2])^4 \\
&= \left(\frac{e^{-2/\theta} - e^{-7/\theta}}{e^{-2/\theta}}\right)^6 \left(\frac{e^{-7/\theta}}{e^{-2/\theta}}\right)^4 \\
&= \left(1 - e^{-5/\theta}\right)^6 \left(e^{-5/\theta}\right)^4 \\
L(u) &= (1-u)^6 \cdot u^4 \qquad u = e^{-5/\theta} \\
\ell(u) &= 6\ln(1-u) + 4\ln(u) \\
\ell'(u) &= \frac{-6}{1-u} + \frac{4}{u} \\
\widehat{u} &= 0.4 = e^{-5/\widehat{\theta}} \\
\widehat{\theta} &= \frac{-5}{\ln(0.4)} = \boxed{5.46}
\end{aligned}$$

Exercise 1

A sample of light bulbs are tested for failure beginning at time $t = 0$. 300 bulbs are still working at time $t = 1$, five fail between time 1 and 10, with failure times at $2, 4, 7, 8$ and 9.
At time 10, the test ends. The time until a lightbulb fails is exponentially distributed with mean θ. Calculate the maximum likelihood estimator of θ.

Exercise 1

A sample of light bulbs are tested for failure beginning at time $t = 0$. 300 bulbs are still working at time $t = 1$, five fail between time 1 and 10, with failure times at $2, 4, 7, 8$ and 9.
At time 10, the test ends. The time until a lightbulb fails is exponentially distributed with mean θ. Calculate the maximum likelihood estimator of θ.

$$\begin{aligned}
\widehat{\theta} &= \frac{\text{sum of observed exposures}}{\#\text{ watched fail}} \\
&= \frac{(2-1)+(4-1)+6+7+8+295\cdot(10-1)}{5} \\
&= \frac{2680}{5} \\
&= \boxed{536}
\end{aligned}$$

Exercise 1 (Continued)

A sample of light bulbs are tested for failure beginning at time $t = 0$. 300 bulbs are still working at time $t = 1$, five fail between time 1 and 10, with failure times at $2, 4, 7, 8$ and 9.
At time 10, the test ends. The time until a lightbulb fails is exponentially distributed with mean θ. Calculate the maximum likelihood estimator of θ.

$$\text{Or: } L(\theta) = \frac{f(2)}{S(1)} \cdot \frac{f(4)}{S(1)} \cdot \dots \cdot \frac{f(9)}{S(1)} \cdot \left[\frac{S(10)}{S(1)}\right]^{295}$$

$$= \left(\frac{1}{e^{-1/\theta}}\right)^{300} \frac{1}{\theta^5} e^{-(2+4+7+8+9)/\theta} \left(e^{-10/\theta}\right)^{295}$$

$$\ell(\theta) = -5\ln(\theta) + \frac{300-(2+4+\dots+9)-10\cdot 295}{\theta}$$

$$0 = \ell'(\theta) = \frac{-5}{\theta} + \frac{2680}{\theta^2} \Rightarrow \widehat{\theta} = \frac{2680}{5} = \boxed{536}$$

Exercise 2

For a group of ten lives, you are given

1. There are five deaths in the interval (0, 1]
2. There are two deaths in the interval (1, 2]
3. There are three deaths in the interval (2, 3]
4. Lifetimes have hazard rate $h(x) = \lambda$

Calculate the maximum likelihood estimate of $e^{-\lambda}$.

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4. Lifetimes have hazard rate $h(x) = \lambda$

Calculate the maximum likelihood estimate of $e^{-\lambda}$.

Let $y = e^{-\lambda}$. $\lambda = 1/\theta$, so $S(x) = e^{-x\lambda}$.

$$\begin{aligned}
L(\lambda) &= \left(1 - e^{-\lambda}\right)^5 \cdot \left(e^{-\lambda} - e^{-2\lambda}\right)^2 \cdot \left(e^{-2\lambda} - e^{-3\lambda}\right)^3 \\
&= \left(1 - e^{-\lambda}\right)^5 \cdot \left(e^{-\lambda}\right)^2 \left(1 - e^{-\lambda}\right)^2 \cdot \left(e^{-\lambda}\right)^{2\cdot 3} \left(1 - e^{-\lambda}\right)^3
\end{aligned}$$

$$L(y) = (1-y)^{10} y^8 \qquad y = e^{-\lambda}$$

$$\ell(y) = 10 \ln(1-y) + 8 \ln(y)$$

$$\ell'(y) = 0 = \frac{-10}{1-y} + \frac{8}{y}$$

$$10y = 8 - 8y \Rightarrow y = \frac{8}{18} = \boxed{\frac{4}{9}}$$