The Infinite Actuary Exam STAM Online Course
C.1.3 Exponentials
Last updated August 15, 2018

1. [4.F01.10] You observe the following five ground-up claims from a data set that is truncated from below at 100:

   125 150 165 175 250

   You fit a ground-up exponential distribution using maximum likelihood estimation. Determine the mean of the fitted distribution.

   A. 73 B. 100 C. 125 D. 156 E. 173

For an exponential, we have

$$\begin{aligned}
\hat{\theta} &= \frac{\text{total payments}}{\#\text{ uncensored observations}} \\
&= \frac{25+50+65+75+150}{5} = \boxed{73}
\end{aligned}$$

Or, using first principles,

$$\begin{aligned}
L(\theta) &= \frac{f(125}{S(100)} \cdot \frac{f(150)}{S(100)} \cdots \frac{f(250)}{S(100)} \\
&= \frac{(1/\theta)e^{-125/\theta}}{e^{-100/\theta}} \cdots \frac{(1/\theta)e^{-250/\theta}}{e^{-100/\theta}} \\
&= \frac{1}{\theta^5} e^{-(25+50+65+75+150)/\theta} \\
\ell(\theta) &= -5\ln(\theta) - \frac{365}{\theta} \\
\ell'(\theta) &= 0 = \frac{-5}{\theta} + \frac{365}{\theta^2} \\
\widehat{\theta} &= \frac{365}{5} = \boxed{73}
\end{aligned}$$

2. [4.F02.23] You are given:

(i) Losses follow an exponential distribution with mean θ.

(ii) A random sample of 20 losses is distributed as follows:

Loss Range	Frequency
$[0, 1000]$	7
$(1000, 2000]$	6
$(2000, \infty)$	7

Calculate the maximum likelihood estimate of θ.

A. Less than 1950
B. At least 1950, but less than 2100
C. At least 2100, but less than 2250
D. At least 2250, but less than 2400
E. At least 2400

Let $p = e^{-1000/\theta}$, so $\mathrm{P}[X \le 1{,}000] = 1-p$, $\mathrm{P}[1{,}000 < X \le 2{,}000] = p(1-p)$ and $\mathrm{P}[X > 2{,}000] = p^2$. Then

$$\begin{aligned}
L(p) &= (1-p)^7[p(1-p)]^6(p^2)^7 \\
&= (1-p)^{13}p^{20} \\
\ell(p) &= 13\ln(1-p) + 20\ln p \\
0 = \ell'(p) &= \frac{-13}{1-p} + \frac{20}{p} \\
p &= \frac{20}{33} = e^{-1000/\theta} \\
\theta &= \boxed{1997}
\end{aligned}$$

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3. [4.F04.26] You are given:

(i) A sample of losses is: 600 700 900

(ii) No information is available about losses of 500 or less.

(iii) Losses are assumed to follow an exponential distribution with mean θ.

Determine the maximum likelihood estimate of θ.

A. 233 B. 400 C. 500 D. 733 E. 1233

For the MLE of an exponential with censored and truncated data,

$$\begin{aligned}
\hat{\theta} &= \frac{\text{sum of observed exposure}}{\text{number of uncensored observations}} \\
&= \frac{(600-500)+(700-500)+(900-500)}{3} \\
&= \boxed{233.3}
\end{aligned}$$

Or from first principles,

$$
\begin{aligned}
L(\theta) &= \frac{f(600}{S(500)} \cdot \frac{f(700)}{S(500)} \cdot \frac{f(900)}{S(500)} \\
&= \frac{(1/\theta)e^{-600/\theta}}{e^{-500/\theta}} \cdot \frac{(1/\theta)e^{-700/\theta}}{e^{-500/\theta}} \cdot \frac{(1/\theta)e^{-900/\theta}}{e^{-500/\theta}} \\
&= \frac{1}{\theta^3} e^{-(600+700+900)/\theta} / e^{-3\cdot 500/\theta} = \frac{1}{\theta^3} e^{-(100+200+400)/\theta} \\
\ell(\theta) &= -3\ln(\theta) - \frac{700}{\theta} \\
\ell'(\theta) &= 0 = \frac{-3}{\theta} + \frac{700}{\theta^2} \\
\widehat{\theta} &= \boxed{\frac{700}{3}}
\end{aligned}
$$

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4. [4.S01.7] You are given a sample of losses from an exponential distribution. However, if a loss is 1000 or greater, it is reported as 1000. The summarized sample is:

Reported Loss	Number	Total Amount
Less than 1000	62	28,140
1000	38	38,000
Total	100	66,140

Determine the maximum likelihood estimate of θ, the mean of the exponential distribution.

A. Less than 650
B. At least 650, but less than 850
C. At least 850, but less than 1050
D. At least 1050, but less than 1250
E. At least 1250

The shortcut approach is that

$$
\begin{aligned}
\widehat{\theta} &= \frac{\text{sum of exposures}}{\#\text{ observed losses}} \\
&= \frac{66,140}{62} = \boxed{1067}
\end{aligned}
$$

The long approach is

$$
\begin{aligned}
L(\theta) &= \left(\prod_{x_i<1000} \frac{1}{\theta} e^{-x_i/\theta} \right) \cdot \left(e^{-1000/\theta} \right)^{38} \\
&= \frac{1}{\theta^{62}} e^{-\sum x_i/\theta} \cdot e^{-38\cdot 1000/\theta} \\
\ell(\theta) &= -62\ln(\theta) - \frac{28,140 + 38,000}{\theta} \\
0 &= \frac{-62}{\theta} + \frac{66,140}{\theta^2} \\
\widehat{\theta} &= \frac{66,140}{62} = \boxed{1067}
\end{aligned}
$$

5. Variant of [4.F04.36] You are given:

   (i) The following is a sample of 15 losses:

   11 22 22 22 36 51 69 69 69 92 92 120 161 161 230

   (ii) $\widehat{H}(x)$ is the maximum likelihood estimate of the cumulative hazard rate function under the assumption that the sample is drawn from an exponential distribution.

   Calculate $\widehat{H}(75)$.

   A. 0.40 B. 0.51 C. 0.60 D. 0.76 E. 0.92

For an exponential, the MLE of θ is $\widehat{\theta} = \overline{X} = 81.8$, and we can use that in finding $\widehat{H}$ by using the fact that MLEs of functions of our parameter are what we get when we plug the MLE into the function. That is,

$$\begin{aligned}\widehat{H}(75) &= -\ln\left[e^{-75/\widehat{\theta}}\right] \\ &= \frac{75}{\widehat{\theta}} = \frac{75}{\overline{X}} \\ &= \frac{75}{81.8} = \boxed{0.917}\end{aligned}$$

6. Variant of [C.F05.5] For a portfolio of policies, you are given:

   (i) There is no deductible and the policy limit varies by policy.

   (ii) A sample of ten claims is:

   350 350 500 500 500^+ 1000 1000^+ 1000^+ 1200 1500

   where the symbol + indicates that the loss exceeds the policy limit.

   (iii) $\widehat{S}(1250)$ is the maximum likelihood estimate of $S(1250)$ under the assumption that the losses follow an exponential distribution.

   Determine $\widehat{S}(1250)$.

   A. 0.21 B. 0.33 C. 0.50 D. 0.67 E. 0.79

$$\begin{aligned}\widehat{\theta} &= \frac{\sum \text{observed exposure time}}{\# \text{ of observed losses}} \\ &= \frac{350 \cdot 2 + 500 \cdot 3 + 1000 \cdot 3 + 1200 + 1500}{10 - 3}\end{aligned}$$

$$
\begin{aligned}
&= \frac{7900}{7} \\
\widehat{S}(1250) &= e^{-1250/\widehat{\theta}} = \boxed{0.33}
\end{aligned}
$$