Probability: Generating Functions

Moment Generating Function

Codes raw moments

Cumulant Generating Function

Codes variance

Probability Generating Function

For counting variables, codes probability mass function

☆ Codes factorial moments

Moment Generating Function

$$M_X(t) = M(t) = Ee^{tX}$$

Example: $f(x) = \lambda e^{-\lambda x}, \quad x > 0$

$$M(t) = Ee^{tX} = \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, dx$$

$$= \lambda \int_0^\infty e^{x(t-\lambda)} \, dx = \frac{\lambda}{t-\lambda} e^{x(t-\lambda)} \Big|_0^\infty$$

If $t < \lambda$, $\quad e^{\infty(t-\lambda)} = 0$,

$$M(t) = \frac{\lambda}{t-\lambda}[0-1] = \frac{\lambda}{\lambda - t}$$

Applying the MGF

$$M(t) = Ee^{tX} \qquad M(0) = Ee^{0} = 1$$

$$M'(t) = EXe^{tX} \qquad M'(0) = EXe^{0} = EX$$

$$M''(t) = E\left(X^2 e^{tX}\right) \qquad M''(0) = E\left(X^2 e^{0}\right) = EX^2$$

$$\vdots$$

$$M^{(k)}(t) = E\left[X^k e^{tX}\right] \qquad M^{(k)}(0) = E\left[X^k e^{0}\right] = E\left(X^k\right)$$

Cumulant Generating Function

$$g(t) = \ln M(t) = \ln\left[E e^{tX}\right]$$

$$g'(0) = \mu = EX$$

$$g''(0) = \mathrm{Var}\, X$$

Examples later

Point: can be a shortcut to find $\mathrm{Var}\, X$

Probability Generating Function

$$P(z) = E z^X \qquad M(t) = E[e^{tX}] = E[(e^t)^X]$$

$$P(e^t) = M(t), \qquad M(\ln z) = P(z)$$

Example: $P[X=0] = \frac{1}{1+\beta}$, $P[X=k] = \frac{1}{1+\beta}\left(\frac{\beta}{1+\beta}\right)^k$

$$E z^X = z^0 P[X=0] + z^1 P[X=1] + \dots + z^k P[X=k] + \dots$$

$$= 1 \cdot \frac{1}{\beta+1} + \frac{1}{\beta+1} \frac{\beta}{\beta+1} \cdot z + \frac{1}{\beta+1} \cdot \left(\frac{\beta}{\beta+1}\right)^2 z^2 + \dots + \frac{1}{\beta+1} \cdot \left(\frac{\beta}{\beta+1}\right)^k z^k + \dots$$

$$= \frac{\frac{1}{\beta+1}}{1 - \frac{\beta}{\beta+1} \cdot z} = \frac{1}{\beta + 1 - \beta z} = \frac{1}{1 + \beta(1-z)}$$

Applications of Probability Generating Functions

$$P(z) = E z^X$$

$$P'(z) = E X z^{X-1}$$

$$P'(1) = E[X \cdot 1] = EX$$

$$P''(z) = E[X(X-1) z^{X-2}]$$

$$P''(1) = E[X(X-1)]$$

$$= E(X^2) - EX$$

$$P'''(z) = E[X(X-1)(X-2) z^{X-3}]$$

$$P'''(1) = E[X(X-1)(X-2)]$$

The moment generating function of X is $M_X(t) = e^{2t^2-5t}$. Find Var$X$.

A. 1 B. 2 C. 3 $\boxed{\text{D. } 4}$ E. 5

$\mathrm{Var}X = E(X^2) - (EX)^2$

$M'(t) = (4t-5)e^{2t^2-5t}$

$M'(0) = -5e^0 = -5$

$EX = -5$

$M''(t) = 4e^{2t^2-5t} + (4t-5)^2 e^{2t^2-5t}$

$M''(0) = 4e^0 + (-5)^2 e^0 = 5^2 + 4$

$\mathrm{Var}X = (5^2+4) - (-5)^2 = 4$

The moment generating function of X is $M_X(t) = e^{2t^2-5t}$. Find Var$X$.

A. 1 B. 2 C. 3 $\boxed{\text{D. 4}}$ E. 5

$g(t) = \ln M(t) = 2t^2 - 5t$

$g'(t) = 4t - 5$

$g''(t) = 4$

$g''(0) = 4 = \text{Var}\,X$

You are given that the probability generating function of a random variable X is

$$P_X(z) = \frac{1}{4-3z} = (4-3z)^{-1}$$

Find the second raw moment of X.

A. 3 B. 9 C. 12 D. 18 $\boxed{\text{E. 21}}$

$P'(1) = EX$, $P''(1) = E[X(X-1)] = E(X^2) - EX$

so $E(X^2) = P''(1) + P'(1)$

$P'(z) = 3(4-3z)^{-2}$

$P'(1) = 3(4-3)^{-2}$

$= 3$

$P''(z) = 18(4-3z)^{-3}$

$P''(1) = 18(1)$

$E(X^2) = 18 + 3 = 21$