Probability: Moments

Definitions

$E[X]$, $E[g(X)]$

Raw moments

Central moments

Coefficients

Coefficient of Variation

Skewness

Kurtosis

Examples

Finding Expected Values

Discrete

$$E[X] = \sum_x x\,p(x)$$

$$E[X^n] = \sum_x x^n p(x)$$

$$E[g(X)] = \sum_x g(x)\,p(x)$$

Continuous

$$E[X] = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x)\,dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\,f(x)\,dx$$

Mixed: sum discrete and continuous parts

Survival Function: $S(x) = 1 - F(x) = P[X > x]$

If $X \geq 0$,

$$E[X] = \int_0^{\infty} S(x)\,dx$$

$$E[g(X)] = \int_0^{\infty} g'(X)\,S(x)\,dx$$

Raw vs. Central Moments

$E[X] = \mu = \mu_X =$ mean $\quad$ 1st moment

$E[X^k] = \mu'_k$ $\quad$ k^{th} raw moment

$E[(X-\mu)^k] = \mu_k$ $\quad$ k^{th} central moment

Note: $Y = aX$

$$E[Y^k] = E[(aX)^k] = a^k E[X^k]$$

k^{th} moment of $Y = a^k \cdot k^{th}$ moment of X

Same hold for central moments

From Raw to Central Moments

$$E[(X-\mu)^2] = \operatorname{Var} X = \sigma^2$$

$$= E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu EX + \mu^2$$

$$= E[X^2] - 2\mu \cdot \mu + \mu^2 = E[X^2] - \mu^2$$

$$E[(X-\mu)^3] = E[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3]$$

$$= E[X^3] - 3\mu E[X^2] + 3\mu^2 \cdot EX - \mu^3$$

$$= E[X^3] - 3\mu E[X^2] + 2\mu^3$$

Coefficient of Variation

$$\text{Var}\, X = \sigma^2 = E(X-\mu)^2 = E(X^2) - \mu^2$$

$$SD(X) = \sigma = \sqrt{\text{Var}\, X} = \text{Standard deviation}$$

$$\text{Coefficient of Variation} = \frac{\sigma}{\mu}$$

Note: if σ^2 is increased, μ unchanged
coeff of var increase

Coeff of Var of cX

$$= \frac{SD(cX)}{E[cX]} = \frac{\cancel{c}\, SD(X)}{\cancel{c}\, E[X]} = \text{Coeff of Var of } X$$

Skewness (aka Skewness Coefficient) $= \dfrac{\mu_3}{\sigma^3}$

Note: if X is symmetric,

Then $E[(X-\mu)^3] = 0$, skewness $= 0$

Note: if $Y = aX + b$

$$E[(Y-\mu_Y)^3] = E[(aX\cancel{+b} - (a\mu_X \cancel{+b}))]^3$$

$$= a^3 \mu_3$$

Likewise $SD(Y) = a\sigma$

$$\text{Skewness}(Y) = \frac{a^3 \mu_{3,X}}{(a\sigma_X)^3} = \frac{\mu_{3,X}}{\sigma_X^3} = \text{Skew}(X)$$

Kurtosis: $= \dfrac{\mu_4}{\sigma^4}$

[3.F01.37] For watches produced by a certain manufacturer:

(i) Lifetimes follow a single-parameter Pareto distribution with $\alpha > 1$ and $\theta = 4$.

(ii) The expected lifetime of a watch is 8 years.

$P[X>6] = 1 - F(6)$

Calculate the probability that the lifetime of a watch is at least 6 years.

$\boxed{\text{A. } 0.44}$ B. 0.50 C. 0.56 D. 0.61 E. 0.67

$E X^k = \frac{\alpha \theta^k}{\alpha - k}$

$8 = \frac{\alpha\, 4}{\alpha - 1}, \quad \alpha = 2$

$F(x) = 1 - \left(\frac{\theta}{x}\right)^\alpha$

$1 - F(6) = 1 - \left(1 - \left(\frac{4}{6}\right)^2\right)$

$= \left(\frac{2}{3}\right)^2 = \frac{4}{9}$

[3.S06.25] Calculate the skewness of a Pareto distribution with $\alpha = 4$ and $\theta = 1,000$.

$$\mu = \frac{\theta}{\alpha-1} = 333.\bar{3}$$

A. Less than 2

B. At least 2, but less than 4

C. At least 4, but less than 6

D. At least 6, but less than 8

E. At least 8

$$\mu_2' = \frac{\theta^2 \cdot 2}{(\alpha-1)(\alpha-2)} = 333,333.\bar{3}$$

$$\mu_3' = \frac{\theta^3 \cdot 6}{(\alpha-1)(\alpha-2)(\alpha-3)} = 10^9$$

$$\mu_3 = E[(X-\mu)^3]$$

$$= E[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3]$$

$$= E[X^3] - 3\mu E[X^2] + 2\mu^3$$

$$= 740,740,741$$

$$\sigma^3 = (EX^2 - \mu^2)^{3/2} = (222,222)^{1.5} = 104,756,560$$

$$\mu_3 / \sigma^3 = 7.07$$

[4.S01.3] You are given the following times of first claim for five randomly selected auto insurance policies observed from time $t = 0$:

$$1 \quad 2 \quad 3 \quad 4 \quad 5$$

Calculate the kurtosis of this sample. $= \dfrac{\mu_4}{\sigma^4} = \dfrac{\frac{34}{5}}{4} = 1.7$

A. 0.0 B. 0.5 $\boxed{\text{C. } 1.7}$ D. 3.4 E. 6.8

x	$p(x)$	$x-\mu$
1	$1/5$	-2
2	$1/5$	-1
3	$1/5$	0
4	$1/5$	1
5	$1/5$	2

$$\sigma^2 = \frac{1}{5}\left[(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2\right] = 2, \quad \sigma^4 = 4$$

$$\mu_4 = \frac{1}{5}\left[(-2)^4 + (-1)^4 + \cdots + (2)^4\right] = \frac{34}{5}$$