Probability: Conditional Moments

Conditional Probability

- Bayes' Theorem
- Law of Total Probability

Conditional Expectation

- Double Expectation

Law of Total Variation

Conditional Probability

$$P[A|B] = \frac{P[AB]}{P[B]} \qquad f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$P[B] \cdot P[A|B] = P[AB]$$

$$\| \qquad //$$

$$P[A] \cdot P[B|A]$$

Bayes' Theorem

$$P[A|B] = \frac{P[A]\,P[B|A]}{P[B]}$$

Law of Total Probability

Suppose $B_1, \ldots, B_n$ are a partition of the probability space, i.e. $B_i \cap B_j = \emptyset$ for all i, j, and $\sum P[B_i] = 1$

Then $P[A] = \sum P[A B_i]$

$$= \sum P[B_i] \cdot P[A \mid B_i]$$

Continuous

$$P[A] = \int_{-\infty}^{\infty} P[A \mid X = x] \, f_X(x) \, dx$$

Conditional Mean (discrete case)

$$E[Y \mid A] = \sum_y y P[Y=y \mid A]$$

$$E[Y \mid X=x] = \sum_y y P[Y=y \mid X=x]$$

$E[Y \mid X]$ is a function of X

$E[Y]$ is a number

Double Expectation

$$E[Y] = E[E[Y \mid X]]$$

Example

You are given that N has a Poisson distribution with mean 2. Find $E[N \mid N \leq 2]$.

$$P[N=n] = e^{-2}\frac{2^n}{n!}$$

$$P[N=0 \mid N \leq 2] = \frac{P[N=0, N \leq 2]}{P[N \leq 2]} = \frac{e^{-2}}{e^{-2}\left[1+2+\frac{2^2}{2}\right]}$$

n	$P[N=n \mid N \leq 2]$
0	$\frac{1}{5}$
1	$\frac{2}{5}$
2	$\frac{2}{5}$

$$E[N \mid N \leq 2]$$

$$= 0 \cdot \frac{1}{5} + 1 \cdot \frac{2}{5} + 2 \cdot \frac{2}{5}$$

$$= \frac{6}{5}$$

Let N be the value rolled by a fair six-sided die. Suppose that I then flip N independent fair coins. What is the expected number of heads? $= H$

A. $1/4$ B. $3/4$ C. $3/2$ $\boxed{\text{D. } 7/4}$ E. $7/2$

Note: if N was fixed, it would be easy, so want to use double expectation.

$$E[H \mid N] = \frac{N}{2}$$

$$EH = E[E[H \mid N]] = E\left[\frac{N}{2}\right]$$

$$= \frac{1}{2} \cdot \frac{7}{2} = \frac{7}{4}$$

Law of Total Variation

Double expectation still works for the second raw moment

$$E[X^2] = E[E[X^2 \mid Y]]$$

Warning! $\mathrm{Var}[X] \neq E[\mathrm{Var}(Y \mid X)]$

$$\mathrm{Var}(X \mid Y) = E\Big([X - \underbrace{E(X \mid Y)}_{\text{will vary as } Y \text{ varies}}]^2 \mid Y\Big)$$

Example: $X = Y$, $Y \sim N(0,1)$

$\mathrm{Var}(X \mid Y) = 0$

But $\mathrm{Var}\, X = \mathrm{Var}\, Y = 1 \neq E[\mathrm{Var}(X \mid Y)] = E[0] = 0$

Law of Total Variation

$$\operatorname{Var} X = \underbrace{E[\operatorname{Var}(X|Y)]}_{\text{expected process variance}} + \underbrace{\operatorname{Var}[E[X|Y]]}_{\text{variance hypothetical mean}}$$

(Based on [110.F86.41]) Let $\mathrm{E}[X \mid Y = y] = 3y$, $\mathrm{Var}[X \mid Y = y] = 2$ and let Y be an exponential random variable with mean 1. What is $\mathrm{Var}[X]$?

A. 3 B. 5 C. 9 $\boxed{\text{D. 11}}$ E. 20

$$\mathrm{Var}\,X = E\left[\mathrm{Var}(X \mid Y=y)\right] + \mathrm{Var}\left[E[X \mid Y=y]\right]$$

$$= E[2] + \mathrm{Var}[3Y]$$

$$= 2 + 3^2\,\mathrm{Var}(Y)$$

$$= 2 + 9 \cdot 1^2$$

$$= 11$$