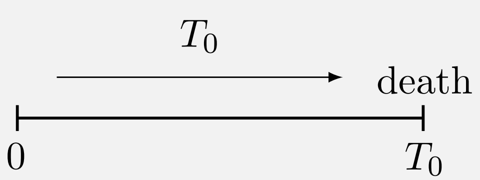
B.1 Survival Distributions and Life Tables

Future Lifetime of a newborn, T_0

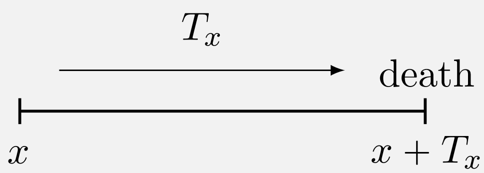
- $T_0 \sim$ RV for future lifetime of newborn
- In other words, the age-at-death RV
- T_0 is continuous
- $T_0 \geqslant 0$



age-at-death RV $\equiv T_0$

Future Lifetime of a person age x, T_x

- (x) - a life age x
- $T_x \sim$ RV for future lifetime of (x)
- T_x is continuous
- $T_x \geqslant 0$



age-at-death RV $\equiv x + T_x$

$$T_0 = x + T_x$$

Cumulative Distribution Function, $F_x(t)$

Probability of (x) dying before age $x + t$.

$$F_x(t) = \Pr(T_x \leqslant t) = \Pr(T_0 \leqslant x+t \,|\, T_0 > x) = \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)}$$

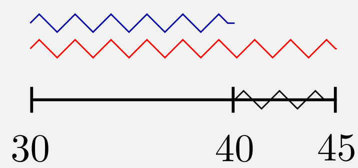
Three properties:

1. $F_x(0) = 0$
2. $\lim_{t \to \infty} F_x(t) = 1$
3. non-decreasing function of t: $F_x(a) \leqslant F_x(b)$ for $a < b$
   $F_x(5) \not> F_x(50)$

Cumulative Distribution Function Example

Example

Express the probability that a (30) dies between ages 40 and 45 using the cumulative distribution function.



$$\Pr(10 < T_{30} < 15) = F_{30}(15) - F_{30}(10)$$

$$= \frac{F_0(45) - F_0(40)}{1 - F_0(30)}$$

Survival Distribution Function, $S_x(t)$

Probability of a (x) attaining age $x + t$. i.e., probability of a (x) not dying before age $x + t$.

$$S_x(t) = \Pr(T_x > t) = 1 - \Pr(T_x \leqslant t) = 1 - F_x(t)$$

$$= \Pr(T_0 > x + t \,|T_0 > x) = \frac{S_0(x+t)}{S_0(x)}$$

Three properties:

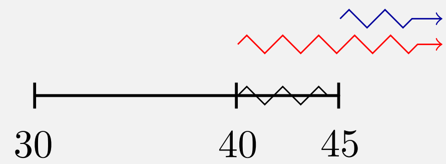
1. $S_x(0) = 1$
2. $\lim_{t\to\infty} S_x(t) = 0$
3. non-increasing function of t: $S_x(a) \geqslant S_x(b)$ for $a < b$

   $S_x(5) \not\leqslant S_x(50)$

Survival Distribution Example

Example

Express the probability that a (30) dies between ages 40 and 45 using the survival function.



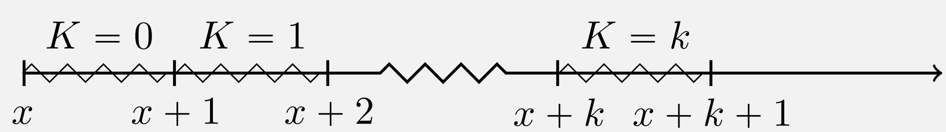
$$\Pr(10 < T_{30} < 15) = S_{30}(10) - S_{30}(15)$$
$$= \frac{S_0(40) - S_0(45)}{S_0(30)}$$

Curtate Future Lifetime of (x), K_x

$$K_x = \lfloor T_x \rfloor$$

where $\lfloor\ \rfloor$ denotes the greatest integer (or floor) function

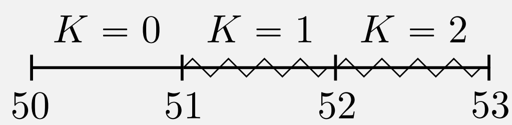
$K_x \sim$ RV for the number of completed future years by (x) prior to death. For integral x, K_x is the number of future birthdays.



Curtate Future Lifetime Example

Example

Write the probability that a 50 year old dies between ages 51 and 53.



$$\Pr(1 \leqslant K_{50} \leqslant 2) = \Pr(K_{50} = 1) + \Pr(K_{50} = 2)$$

Exercise

You are given a survival function $S_0(x) = 1 - 0.01x$ for $0 \leqslant x \leqslant 100$.

Determine the median future lifetime of a life aged 10.

Let m be the median.

$$\Pr[T_{10} \leqslant m] = 0.5$$

$$\Pr[T_{10} > m] = 1 - 0.5 = 0.5$$

$$S_{10}(m) = 0.5$$

$$\frac{S_0(m+10)}{S_0(10)} = 0.5$$

$$\frac{1 - 0.01(m+10)}{1 - 0.01(10)} = 0.5$$

$$m = \boxed{45}$$

B.1 Survival Distributions and Life Tables

B.1.1 Probability Functions

B.1.2 Actuarial Notation for Probabilities

- ${}_tp_x$
- ${}_tq_x$
- ${}_{t|u}q_x$
- Revisit Curtate Future Lifetime
- Summary of Actuarial Notation
- Exercises

B.1.3 Life Tables

${}_tp_x$

Definition (${}_tp_x$)

Probability that (x) will attain age $x+t$.

$$\begin{aligned} {}_tp_x &= \Pr(T_x > t) \\ &= S_x(t) \end{aligned}$$

If $t=1$, prefix omitted

$$p_x = \text{probability } (x) \text{ will attain age } x+1$$

$${}_{t+u}p_x = {}_tp_x \cdot {}_up_{x+t}$$

${}_tq_x$

Definition (${}_tq_x$)

Probability that (x) dies within t years.

$$\begin{aligned} {}_tq_x &= \Pr(T_x \leqslant t) \\ &= F_x(t) \end{aligned}$$

$${}_tq_x = 1 - {}_tp_x$$

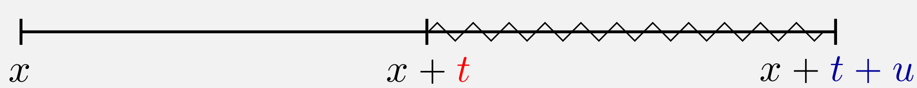
If $t = 1$, prefix omitted

$$\begin{aligned} q_x &= \text{probability } (x) \text{ dies in 1 year} \\ &\quad \text{mortality rate at age } x \end{aligned}$$

${}_{t|u}q_x$

Definition (${}_{t|u}q_x$)

Probability that (x) will survive t years and die within the following u years.

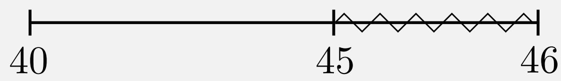


$$\begin{aligned} {}_{t|u}q_x &= \Pr(t < T_x \leqslant t + u) = S_x(t) - S_x(t+u) \\ &= {}_tp_x - {}_{t+u}p_x \\ &= {}_{t+u}q_x - {}_tq_x \\ &= {}_tp_x \cdot {}_uq_{x+t} \end{aligned}$$

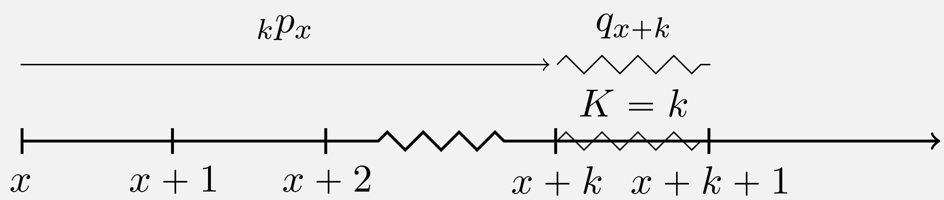
${}_{t|u}q_x$ cont.

If $u = 1$, prefix omitted

${}_{5|}q_{40}$



Revisit Curtate Future Lifetime



$$\Pr(K_x = k) = {}_kp_x \cdot q_{x+k} = {}_{k|}q_x$$

$\therefore$ ${}_{k|}q_x$ is PDF for curtate future lifetime for (x)

$$\sum_{k=0}^{\infty} {}_{k|}q_x = 1$$

$$F_{K_x}(y) = \sum_{h=0}^{z} {}_{h|}q_x \quad \text{where } z \text{ is the greatest integer in } y$$

Summary of Actuarial Notation

1. ${}_tp_x$
   - probability (x) survives at least t years
   - $S_x(t)$, survival function for T_x
2. ${}_tq_x$
   - probability (x) dies before age $x+t$
   - $F_x(t)$, cumulative distribution function for T_x
3. ${}_{t|u}q_x$
   - probability (x) survives t years and dies in the following u years
   - ${}_{t|}q_x$ is the probability density function for K_x

Exercise

You are given ${}_{1|}q_{x+1} = 0.095$, ${}_{2|}q_{x+1} = 0.171$ and $q_{x+3} = 0.200$. Calculate $q_{x+1} + q_{x+2}$.

Step 1 - Find ${}_2p_{x+1}$

$$
\begin{aligned}
{}_{2|}q_{x+1} &= {}_2p_{x+1} \cdot q_{x+3} \\
0.171 &= {}_2p_{x+1} \cdot 0.2 \\
{}_2p_{x+1} &= 0.855
\end{aligned}
$$

Step 2 - Find q_{x+1}

$$
\begin{aligned}
{}_{1|}q_{x+1} &= p_{x+1} - {}_2p_{x+1} \\
0.095 &= p_{x+1} - 0.855 \\
p_{x+1} &= 0.95 \\
q_{x+1} &= 0.05
\end{aligned}
$$

Step 3 - Find q_{x+2}

$$
\begin{aligned}
{}_2p_{x+1} &= 0.855 \\
p_{x+1} \cdot p_{x+2} &= 0.855 \\
0.95 \cdot p_{x+2} &= 0.855 \\
p_{x+2} &= 0.9 \\
q_{x+2} &= 0.1
\end{aligned}
$$

$$q_{x+1} + q_{x+2} = 0.05 + 0.1 = \boxed{0.15}$$

Exercise - Problem Solving Technique

Given ${}_{1|}q_{x+1}$, ${}_{2|}q_{x+1}$ and q_{x+3}, find q_{x+1} and q_{x+2}.

Write out every relationship for the given information that you can think of:

- $$\begin{aligned} {}_{1|}q_{x+1} &= p_{x+1} \cdot q_{x+2} \\ &= p_{x+1} - {}_2p_{x+1} \\ &= {}_2q_{x+1} - q_{x+1} \end{aligned}$$ 3 - finally find q_{x+2}; 2 - plug into here to find q_{x+1}

- $$\begin{aligned} {}_{2|}q_{x+1} &= {}_2p_{x+1} \cdot q_{x+3} \\ &= {}_2p_{x+1} - {}_3p_{x+1} \\ &= {}_3q_{x+1} - {}_2q_{x+1} \end{aligned}$$ 1 - start here b/c one unknown

B.1 Survival Distributions and Life Tables

Life Tables Notation

- l_x - expected number of survivors at (x)
- ${}_n d_x$ - expected number of deaths between ages x and $x+n$
- d_x - expected number of deaths between ages x and $x+1$

Life Table Example

x	l_x	
0	81	radix
1	64	
2	49	
3	36	
4	25	
5	16	
6	9	
7	4	
8	1	
9	0	ω

$$l_x = l_0 \cdot S_0(x)$$

$$S_0(4) = \frac{25}{81}$$

$$p_x = \frac{l_{x+1}}{l_x} \qquad q_x = \frac{d_x}{l_x} = \frac{l_x - l_{x+1}}{l_x}$$

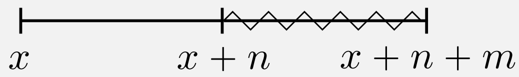
$$_np_x = \frac{l_{x+n}}{l_x} \qquad _nq_x = \frac{_nd_x}{l_x} = \frac{l_x - l_{x+n}}{l_x}$$

$$_2p_5 = \frac{4}{16} \qquad _3q_1 = \frac{64-25}{64} = \frac{39}{64}$$

Life Table Example cont.

x	l_x	
0	81	radix
1	64	
2	49	
3	36	
4	25	
5	16	
6	9	
7	4	
8	1	
9	0	ω

$$ {}_{n|m}q_x = \frac{l_{x+n} - l_{x+n+m}}{l_x} $$



$$ {}_{2|}q_3 = \frac{l_5 - l_6}{l_3} = \frac{16 - 9}{36} = \frac{7}{36} $$

Table built from

$$ S_0(x) = \left(\frac{9-x}{9}\right)^2 $$

$$ l_x = 81\left(\frac{9-x}{9}\right)^2 $$

${}_{2.5}p_3$ cannot be determined from table

Illustrative Life Table (ILT)

Illustrative Life Table: Basic Functions and Single Benefit Premiums at i = 0.06

x	l_x	$1000q_x$	$\ddot{a}_x$	$1000A_x$	$1000(^2A_x)$	$1000\,{}_5E_x$	$1000\,{}_{10}E_x$	$1000\,{}_{20}E_x$	x
0	10,000,000	20.42	16.8010	49.00	25.92	728.54	541.95	299.89	0
5	9,749,503	0.98	17.0379	35.59	8.45	743.89	553.48	305.90	5
10	9,705,588	0.85	16.9119	42.72	9.37	744.04	553.34	305.24	10
15	9,663,731	0.91	16.7384	52.55	11.33	743.71	552.69	303.96	15
20	9,617,802	1.03	16.5133	65.28	14.30	743.16	551.64	301.93	20
21	9,607,896	1.06	16.4611	68.24	15.06	743.01	551.36	301.40	21
22	9,597,695	1.10	16.4061	71.35	15.87	742.86	551.06	300.82	22
23	9,587,169	1.13	16.3484	74.62	16.76	742.68	550.73	300.19	23
24	9,576,288	1.18	16.2878	78.05	17.71	742.49	550.36	299.49	24
25	9,565,017	1.22	16.2242	81.65	18.75	742.29	549.97	298.73	25
26	9,553,319	1.27	16.1574	85.43	19.87	742.06	549.53	297.90	26
27	9,541,153	1.33	16.0873	89.40	21.07	741.81	549.05	297.00	27
28	9,528,475	1.39	16.0139	93.56	22.38	741.54	548.53	296.01	28
29	9,515,235	1.46	15.9368	97.92	23.79	741.24	547.96	294.92	29
30	9,501,381	1.53	15.8561	102.48	25.31	740.91	547.33	293.74	30
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	
106	727	597.83	1.5685	911.22	832.53	2.99	0.00	0.00	106
107	292	631.64	1.4984	915.19	839.46	1.76	0.00	0.00	107
108	108	665.45	1.4341	918.82	845.84	0.98	0.00	0.00	108
109	36	698.97	1.3755	922.14	851.69	0.52	0.00	0.00	109
110	11	731.87	1.3223	925.15	857.04	0.26	0.00	0.00	110

Exercise 1

(same exercise from lesson B.1.2, but solved using life table now)

You are given ${}_{1|}q_{x+1} = 0.095$, ${}_{2|}q_{x+1} = 0.171$ and $q_{x+3} = 0.200$. Calculate $q_{x+1} + q_{x+2}$.

y	l_y	d_y
$x+1$	1000	$1000 - 950 = 50$
$x+2$	$855 + 95 = 950$	$1000 \times 0.095 = 95$
$x+3$	$171/0.2 = 855$	$1000 \times 0.171 = 171$

$$q_{x+1} + q_{x+2} = \frac{50}{1000} + \frac{95}{950} = \boxed{0.15}$$

Exercise 2

You are given the following:

A. The probability that a person age 20 will survive 30 years is 0.7.
B. The probability that a person age 45 will die within 5 years and that another person age 40 will survive 5 years is 0.0475.
C. The probability that a person age 20 will survive 20 years and that another person age 40 will die within 5 years is 0.04.

Calculate the probability that a person age 20 will survive 25 years.

Restate the problem using actuarial notation:

A. ${}_{30}p_{20} = 0.7$
B. ${}_{5|5}q_{40} = 0.0475$
C. ${}_{20|5}q_{20} = 0.04$

Calculate ${}_{25}p_{20}$.

Exercise 2 cont.

Given:

A. ${}_{30}p_{20} = 0.7$

B. ${}_{5|5}q_{40} = 0.0475$

C. ${}_{20|5}q_{20} = 0.04$

Calculate ${}_{25}p_{20}$.

$$\star \quad l_{40} - 40 = 700 + l_{40}(0.0475)$$

$$l_{40} = 777$$

$$l_{45} = 777 - 40 = 737$$

$${}_{25}p_{20} = \frac{737}{1000} = \boxed{0.737}$$

x	l_x	${}_5d_x$
20	1000	
$\vdots$		
40	l_{40}	$1000 \times 0.04 = 40$
45	$\star$	$l_{40}(0.0475)$
50	$1000 \times 0.7 = 700$	