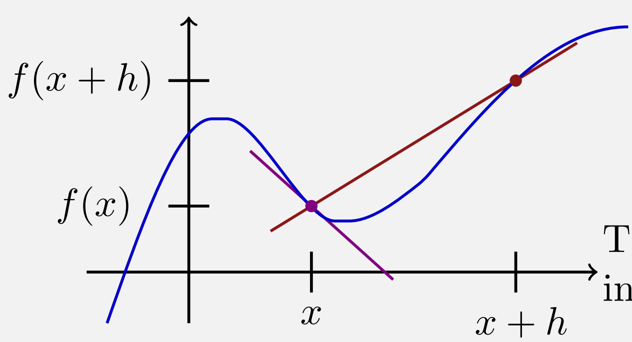
B.0 One Dimensional Calculus - Outline

Definition of Derivative (Background Only)



The average rate of change from x to $x+h$ is

$$= \frac{\text{total change}}{\text{length of interval}}$$

$$= \frac{f(x+h)-f(x)}{h}$$

The derivative is the instantaneous rate of change

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \frac{f(x+dx)-f(x)}{dx} = \frac{df}{dx}$$

Example

The definition is typically cumbersome to use.
For example,

$$\begin{aligned}\frac{d}{dx}x^2 &= \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h\to 0}\frac{\left(x^2 + 2xh + h^2\right) - x^2}{h} \\ &= \lim_{h\to 0}\frac{2xh + h^2}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x\end{aligned}$$

Instead of always doing this, in practice people use a smaller number of key formulas.

Basic Formulas

$$\begin{aligned}\frac{d}{dx}f(x) &= f'(x) \\ \frac{d}{dx}a &= 0 \text{ for any constant } a \\ \frac{d}{dx}x^n &= nx^{n-1} \\ \frac{d}{dx}[\ln(x)] &= \frac{1}{x} \\ \frac{d}{dx}e^x &= e^x \\ \frac{d}{dx}[f(x)+g(x)] &= f'(x)+g'(x) \\ \frac{d}{dx}[c\,f(x)] &= c\frac{d}{dx}f(x) = cf'(x)\end{aligned}$$

Basic Formulas: Examples

$$\begin{aligned}
\frac{d}{dx}x^3 &= 3x^2 \\
\frac{d}{dy}3y^2 &= 3 \cdot 2y^1 = 6y \\
\frac{d}{dt}\left(5e^t + 3t^4\right) &= 5e^t + 3 \cdot 4t^3 \\
\frac{d}{ds}\frac{2}{s^3} &= \frac{d}{ds}2s^{-3} \\
&= -6s^{-4} \\
&= \frac{-6}{s^4}
\end{aligned}$$

Chain Rule

Theorem (Chain Rule)

If f and u are differentiable functions,

$$\frac{d}{dx}[f(u)] = f'(u) \cdot \frac{du}{dx}$$

Examples

$$\begin{aligned}
\frac{d}{dx}u^n &= nu^{n-1}\frac{du}{dx} \\
\frac{d}{dx}e^u &= e^u \cdot \frac{du}{dx}
\end{aligned}$$

Chain Rule: Examples

Suppose we want to find $\frac{d}{dt}\exp\left[-2t+t^2\right]$. Let $u=-2t+t^2$.

$$\begin{aligned}
\frac{d}{dt}\exp\left[-2t+t^2\right] &= \frac{d}{dt}\exp\left[u\right] \\
&= \exp\left[u\right]\cdot\frac{du}{dx} \\
&= \exp\left[-2t+t^2\right]\cdot(-2+2t) \\
\frac{d}{dx}\left(2x^2+5x+3\right)^5 &= 5\left(2x^2+5x+3\right)^4\cdot(4x+5) \\
\frac{d}{dt}\exp\left[5e^t-5+3t\right] &= \left(\exp\left[5e^t-5+3t\right]\right)\cdot\left(5e^t+3\right)
\end{aligned}$$

Chain Rule: Examples

Suppose we want to find $\frac{d}{dx}2^x$. We know how to find $\frac{d}{dx}e^x$, so let's rewrite 2^x in terms of e^x.

$$2=e^{\ln(2)}, 2^x=\left(e^{\ln(2)}\right)^x=e^{x\ln(2)}$$

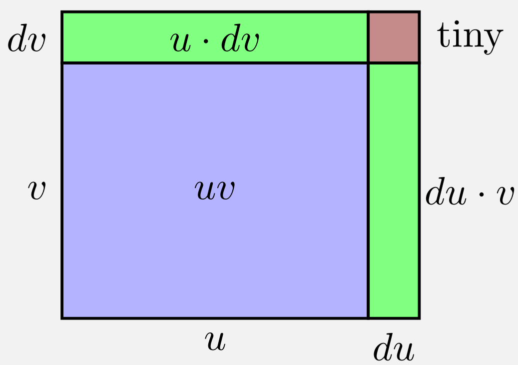
$$\frac{d}{dx}2^x=\frac{d}{dx}e^{x\ln(2)}=\ln(2)\cdot e^{x\ln(2)}=(\ln 2)\cdot 2^x$$

More generally, for any a,

$$\begin{aligned}
\frac{d}{dx}a^x &= (\ln a)\cdot a^x \\
\frac{d}{dx}2^{x^3-3x} &= \frac{d}{dx}2^u \quad u=x^3-3x \\
&= 2^u\cdot\ln(2)\cdot\frac{du}{dx} \\
&= 2^{\left(x^3-3x\right)}\cdot\ln(2)\cdot\left[3x^2-3\right]
\end{aligned}$$

Product Rule

What about the derivatives of products or quotients of two functions?



$$\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + \frac{du}{dx} \cdot v$$
$$(uv)' = u \cdot v' + u' \cdot v$$

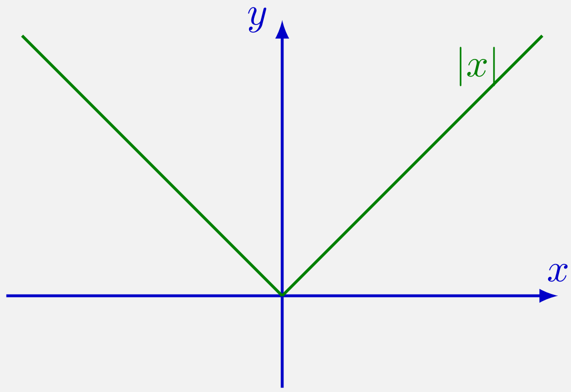
Quotients can be done by rewriting them as products

$$\begin{aligned} \frac{d}{dx}\left[u \cdot \frac{1}{v}\right] &= u \cdot \frac{d}{dx}\frac{1}{v} + u' \cdot \frac{1}{v} \\ &= \frac{-u\,v'}{v^2} + \frac{u'}{v} \\ &= \frac{-uv' + u'v}{v^2} \end{aligned}$$

Product Rule Examples

$$\begin{aligned} \frac{d}{dx}\left[x^2 e^{3x}\right] &= x^2 \cdot 3e^{3x} + 2x \cdot e^{3x} \\ \frac{d}{dx}\frac{e^{-x}}{x^3} &= \frac{d}{dx}\left[e^{-x} \cdot x^{-3}\right] \\ &= e^{-x} \cdot \frac{-3}{x^4} + (-e^{-x}) \cdot \frac{1}{x^3} \\ \frac{d}{dx}\left[\left(x^2+3x+5\right)^3 \cdot e^{4x}\right] &= \left(x^2+3x+5\right)^3 \cdot \frac{d}{dx}e^{4x} \\ &\quad + \frac{d}{dx}\left[\left(x^2+3x+5\right)^3\right] \cdot e^{4x} \\ &= \left(x^2+3x+5\right)^3 \cdot 4e^{4x} \\ &\quad + 3 \cdot (x^2+3x+5)^2 \cdot (2x+3) \cdot e^{4x} \end{aligned}$$

Absolute Values



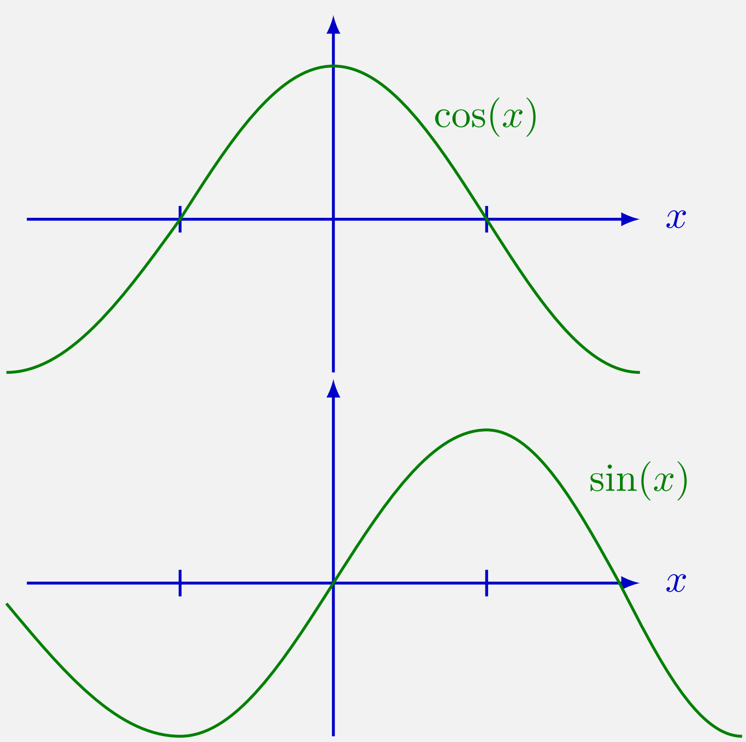
$$|x| = x \quad \text{if } x \geq 0$$

$$|x| = -x \quad \text{if } x < 0$$

$$\frac{d}{dx}|x| = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

Note that $\frac{d|x|}{dx}$ is undefined if $x = 0$.

Sines and Cosines



$$\frac{d}{dx}\cos(x) = -\sin(x)$$

$$\frac{d}{dx}\sin(x) = \cos(x)$$

Further Examples

$$\frac{d}{dx}\,\frac{2x+5}{x^2-3x+4} = (2x+5)\frac{(-1)(2x-3)}{(x^2-3x+4)^2} + \frac{2}{x^2-3x+4}$$

$$\frac{d}{dx}(x+2)e^{x^2-5x} = (x+2)(2x-5)e^{x^2-5x} + 1\cdot e^{x^2-5x}$$

$$\frac{d}{dx}x^2e^{-3x^2} = x^2(-6x)e^{-3x^2} + 2x\cdot e^{-3x^2}$$

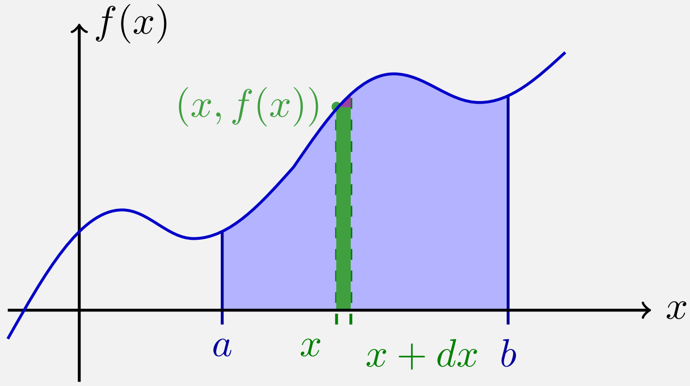
Further Examples

$$\begin{aligned}
\frac{d}{dx}\sin|x+2| &= \frac{d}{dx}\sin(x+2) \quad \text{if } x+2>0\\
&= \cos(x+2) \qquad\qquad x>-2\\
\frac{d}{dx}\sin|x+2| &= \frac{d}{dx}\sin(-x-2) \quad \text{if } x+2<0\\
&= -\cos(-x-2) \qquad\quad x<-2
\end{aligned}$$

Key point: We get two cases based on whether or not what is inside the absolute value is positive. If $x+2>0$ then what is inside the absolute value is positive, so $|x+2| = x+2$, while if $x+2<0$ then what is inside the absolute value is negative so $|x+2| = -x-2$.

B.0 One Dimensional Calculus - Outline

Definition of an Integral

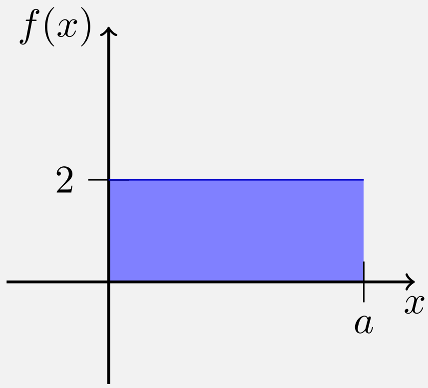


$$\int_a^b f(x)\,dx = \text{area under curve}$$

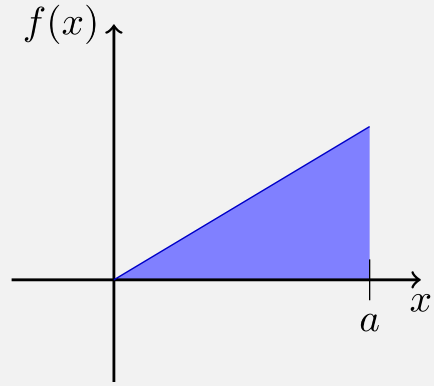
In some sense, $f(x)\,dx$ is the area of an infinitely thin rectangle and the integral says that the area under the curve is the sum of the areas of infinitely many of these thin rectangles.

Geometric Examples

Often we can use geometry to find the integral/area under the curve.



$$\int_0^a 2dx = 2a$$



$$\int_0^a xdx = \frac{a^2}{2}$$

Geometric Examples

In that example,

1. The integral of a constant was a linear function.
2. The integral of a line was a quadratic function.

So in these two examples, when we integrated the power of a polynomial increased by 1.

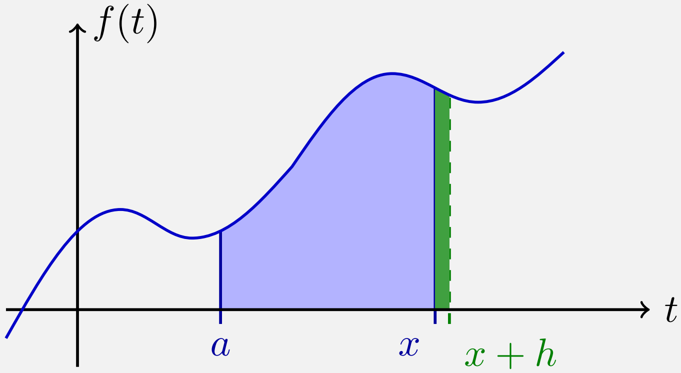
When we differentiate,

1. The derivative of a linear function is a constant.
2. The derivative of a quadratic function is linear.

More generally, when we differentiate the power of a polynomial decreases by 1. That is the opposite of when we integrate.

Hmmm.......isn't that an interesting coincidence?

The Fundamental Theorem of Calculus



$$\frac{d}{dx}F(x) = \frac{d}{dx}\int_a^x f(t)\,dt = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0}\frac{f(x)\cdot h}{h} = f(x)$$

The Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Generalization: If v and u are functions,

$$\frac{d}{dx}\int_a^v f(t)\,dt = f(v)\,\frac{dv}{dx}\frac{d}{dx}\int_u^v f(t)\,dt \quad = f(v)\,\frac{dv}{dx} - f(u)\,\frac{du}{dx}$$

In words, the Fundamental Theorem of Calculus says that derivatives and integrals are inverse operations.

Evaluating Integrals

The Fundamental Theorem of Calculus says that derivatives and integrals are inverse operations. To find the integral of $f(x)$, we need to find a function whose derivative is $f(x)$.

Examples

$$\int_0^5 x\,dx = \frac{1}{2}x^2\,\Big|_0^5 = \frac{5^2}{2} - \frac{0^2}{2} = \frac{25}{2}$$

$$\begin{aligned}\int\limits_a^b x^n\,dx &= \frac{1}{n+1}\cdot x^{n+1}\,\Big|_a^b \\ &= \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}\end{aligned}$$

Common Formulas

$$\int a\,dx = ax + C$$

$$\int x\,dx = \frac{x^2}{2} + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{for } n \neq -1$$

$$\int e^{bx}\,dx = \frac{1}{b}e^{bx} + C$$

$$\begin{aligned}\int a^x\,dx &= \int e^{x\ln a}\,dx \\ &= \frac{1}{\ln a}a^x + C\end{aligned}$$

Examples

$$\int_{-2}^{5} 3x^4\,dx = 3 \cdot \frac{x^5}{5}\bigg|_{-2}^{5}$$
$$= 3 \cdot \frac{5^5}{5} - 3 \cdot \frac{(-2)^5}{5}$$
$$\int_{2}^{\infty} \frac{3}{x^4}\,dx = \frac{3}{x^3} \cdot \frac{1}{-3}\bigg|_{2}^{\infty}$$
$$= \frac{-1}{\infty^3} - \frac{-1}{2^3}$$
$$= 0 + \frac{1}{8} = \frac{1}{8}$$

Substitution

When we differentiate, we often have nested functions and need to use the chain rule. For example,

$$\frac{d}{dx}e^{x^2} = e^{x^2} \cdot 2x$$
$$\frac{d}{dx}(x^2+3)^5 = 5(x^2+3)^4 \cdot 2x$$

In both those examples, the $2x$ factor comes from the chain rule. Often when we are doing integration, we will have a term that we need to somehow recognize as a chain rule factor. If we can do that, we can do a substitution to do the chain rule "backwards."

Substitution

Suppose we want to integrate $2x \cdot e^{\left(x^2\right)}$. Let $u = x^2$. Then $\frac{du}{dx} = 2x$ so $du = 2x\,dx$ and we get

$$\int_{x=a}^{x=b} 2x\, e^{\left(x^2\right)}\, dx = \int_{u=a^2}^{u=b^2} e^u\, du$$
$$= e^u \Big|_{u=a^2}^{u=b^2} = e^{\left(x^2\right)} \Big|_{x=a}^{x=b}$$
$$= e^{b^2} - e^{a^2}$$

Note the limits! Either we convert back to x at the end, or we change the limits to be in terms of u.

Substitution Examples

$$\int_2^\infty x e^{-2x^2}\, dx \qquad u = 2x^2 \quad du = 4x\, dx$$
$$x = 2 \quad u = 2 \cdot 2^2 = 8$$
$$x = \infty \quad u = 2 \cdot \infty^2 = \infty$$

$$= \int_8^\infty e^{-u} \cdot \frac{du}{4}$$

$$= \frac{1}{4} \cdot (-1) \cdot e^{-u} \Big|_8^\infty = 0 - \frac{-1}{4} e^{-8}$$

$$= \frac{1}{4} e^{-8}$$

Other Formulas

$$\int \frac{dx}{x} = \ln x + C$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

$$\int \big[f(x)+g(x)\big]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int \cos\, x\,dx = \sin\, x + C \quad \text{because } \frac{d}{dx}\sin x = \cos x$$

$$\int \sin\, x\,dx = -\cos\, x + C \quad \text{because } \frac{d}{dx}\cos x = -\sin x$$

Examples

$$\begin{aligned} u &= x+5 \qquad & x=2, u=2+5=7 \\ du &= dx \qquad & x=5, u=5+5=10 \end{aligned}$$

$$\begin{aligned} \int_2^5 \frac{3x}{(x+5)^2}\,dx &= \int_7^{10} \frac{3(u-5)}{u^2} du \\ &= \int_7^{10} \frac{3}{u} - \frac{15}{u^2}\,du \\ &= \left(3\ln u + \frac{15}{u}\right)\Bigg|_7^{10} \\ &= \left(3\ln 10 + \frac{15}{10}\right) - \left(3\ln 7 + \frac{15}{7}\right) \end{aligned}$$

Examples

$$\int_0^\pi (1+\cos t)dt = t + \sin t \Big|_0^\pi$$

$$= (\pi + 0) - (0+0) = \pi$$

$$\int_{-2}^{5} |x|\, dx = \int_{-2}^{0} -x\, dx + \int_{0}^{5} x\, dx$$

$$= \frac{-x^2}{2}\Big|_{-2}^{0} + \frac{x^2}{2}\Big|_0^5 = \frac{(-2)^2}{2} + \frac{25}{2} = \frac{29}{2}$$

$$\frac{d}{dx}\int_{-2x}^{x^3} e^{5t-5}dt = e^{5x^3-5}\cdot 3x^2 - e^{5(-2x)-5}(-2)$$

B.0 One Dimensional Calculus - Outline

Integration By Parts

Integration by parts is doing the product rule backwards.

$$\frac{d}{dx}uv = u \cdot \frac{dv}{dx} + \frac{du}{dx} \cdot v$$

$$\int d\,(uv) = uv = \int u\,dv + \int v\,du$$

$$\int u \cdot dv = uv - \int v\,du$$

Integration By Parts

Example

$$\int x\, e^x\, dx = \int u\, dv$$

$$u = x \qquad dv = e^x\, dx$$

$$du = dx \qquad v = e^x$$

so

$$\int x\, e^x\, dx = uv - \int vdu$$

$$= xe^x - \int e^x\, dx$$

$$= xe^x - e^x + C$$

Logarithms

You can use integration by parts to handle functions whose derivatives are easier to find than their integrals.

$$\int \ln x \, dx = \int u \, dv$$

$$u = \ln x \qquad dv = dx$$

$$du = \frac{dx}{x} \qquad v = x$$

so

$$\begin{aligned} \int \ln x \, dx &= uv - \int v du \\ &= (\ln x)x - \int x \frac{dx}{x} \\ &= x \ln x - x + C \end{aligned}$$

Logarithms

$$\int x \ln x \, dx = \int u \, dv$$

$$u = \ln x \qquad dv = x \, dx$$

$$du = \frac{dx}{x} \qquad v = \frac{x^2}{2}$$

so

$$\begin{aligned} \int x \ln x \, dx &= uv - \int v du \\ &= \frac{x^2}{2} \ln x - \int \frac{x^2}{2x} dx \\ &= \frac{x^2}{2} \ln x - \frac{x^2}{4} + C \end{aligned}$$

Which Part to Use

How do we choose u and dv? We need dv to be something that is easy to integrate and we need u to be something that is easy to differentiate.
Ideally we also want u to become simpler when you differentiate.

Easy to Differentiate		Easy to Integrate
Logs	Polynomials	Exponentials

Integration by parts

Iterated Parts

$$\int x^2 e^{2x}\,dx \qquad \text{Let} \quad u = x^2 \qquad dv = e^{2x}\,dx$$

$$du = 2x\,dx \qquad v = \frac{1}{2}\,e^{2x}$$

$$= x^2 \cdot \frac{1}{2}\,e^{2x} - \int \frac{1}{2}\,e^{2x}\,2x\,dx$$

$$= \frac{1}{2}\,x^2\,e^{2x} - \int x\,e^{2x}\,dx$$

And to find this, we have to repeat integration by parts.

Tabular integration

Tabular integration is a way to organize our work when doing repeated integration by parts. To integrate $x^2 e^{2x}$,

Derivative column		Integral column
x^2	$+$	e^{2x}
$2x$	$-$	$\frac{1}{2} e^{2x}$
2	$+$	$\frac{1}{4} e^{2x}$
0	$-$	$\frac{1}{8} e^{2x}$

$$\int x^2 e^{2x}\, dx = x^2 \cdot \frac{1}{2} e^{2x} - 2x \cdot \frac{1}{4} \cdot e^{2x} + 2 \cdot \frac{1}{8} e^{2x} - 0$$

The Gamma trick

If we have a definite integral from 0 to infinity, we often can skip using integration by parts.
If $b > 0$ and a is an integer, then

$$\int_0^\infty x^a \cdot e^{-bx} dx = \frac{a!}{b^{a+1}}$$

Example

$$\int_0^\infty x^2 e^{-2x} dx = \frac{2!}{2^{2+1}} = \frac{1}{4}$$

$-b = -2 \qquad b = 2$

$a = 2$

The Gamma trick

An example both ways: $\int_0^\infty 4x^2\, e^{-x/3} dx$

Derivative column		Integral column
$4x^2$		$e^{-x/3}$
$8x$	$+$	$-3\, e^{-x/3}$
8	$-$	$9\, e^{-x/3}$
0	$+$	$-27\, e^{-x/3}$

So $\int 4x^2 e^{-x/3} dx$ is

$$(4x^2)(-3\, e^{-x/3}) - (8x)(9\, e^{-x/3}) + 8 \cdot (-27\, e^{-x/3})$$

$$= (-12x^2 - 72x - 8 \cdot 27)\, e^{-x/3}$$

The Gamma trick

$$\begin{aligned}\int_0^\infty 4x^2 e^{-x/3} &= (-12x^2 - 72x - 8 \cdot 27) e^{-x/3}\Big|_0^\infty \\ &= 8 \cdot 27\end{aligned}$$

or we can let $b = 1/3$ and $a = 2$ in our formula to get

$$\begin{aligned}\int_0^\infty x^a e^{-bx} &= \frac{a!}{b^{a+1}} \\ \int_0^\infty x^2 e^{-x/3} &= \frac{2!}{(1/3)^{2+1}} = 2 \cdot 27 \\ \int_0^\infty 4x^2 e^{-x/3} &= 4 \cdot \frac{2!}{\left(\frac{1}{3}\right)^{2+1}} = 8 \cdot 27\end{aligned}$$

Other Definite Integrals

$$\int_1^\infty 4x^2\, e^{-x/3} \qquad \text{We want } u = 0 \quad \text{when } x = 1$$

$$u = x - 1, \quad x = u + 1$$

$$= \int_0^\infty 4(u+1)^2\, e^{(-u-1)/3} du$$

$$= 4\, e^{-1/3} \int_0^\infty \left(u^2 + 2u + 1\right) e^{-u/3}\, du$$

and now we plug into $\displaystyle\int_0^\infty x^a\, e^{-bx} = \frac{a!}{b^{a+1}}$

$$= 4\, e^{-1/3} \left[\frac{2!}{\left(\frac{1}{3}\right)^3} + 2 \cdot \frac{1}{\left(\frac{1}{3}\right)^2} + \frac{1}{\frac{1}{3}} \right]$$

The Gamma trick

Idea of Proof:

Let $u = x^a$ and $dv = e^{-bx} dx$

$$\int_0^\infty x^a e^{-bx} dx = x^a \cdot \frac{-1}{b} e^{-bx} \Big|_0^\infty - \int_0^\infty a x^{a-1} \left(\frac{-1}{b}\right) e^{-bx} dx$$

$$= 0 - 0 + \frac{a}{b} \int_0^\infty x^{a-1} e^{-bx} dx$$

$$= \frac{a}{b} \cdot \frac{(a-1)!}{b^{a-1+1}}$$

$$= \frac{a!}{b^{a+1}}$$

It also is related to $\mathrm{E}\left[X^a\right]$ when X is an exponential random variable as well as the density of a Gamma random variable.