A.2 Key Continuous Distributions - Outline

Exponential Distribution

If X is exponential with parameter θ then

$$\begin{aligned}
\mathrm{P}[X > x] &= S(x) = e^{-x/\theta} \quad x > 0 \\
\mathrm{E}[X] &= \int_0^\infty e^{-x/\theta} dx = -\theta e^{-x/\theta}\Big|_0^\infty = \theta \\
f(x) &= \frac{1}{\theta} e^{-x/\theta} = \lambda e^{-\lambda x} \quad \text{where } \lambda = 1/\theta \\
h(x) &= \frac{1}{\theta} = \lambda = \text{“rate”}
\end{aligned}$$

Higher moments:

$$\begin{aligned}
\mathrm{E}\left[X^2\right] &= 2\theta^2 \\
\mathrm{Var}[X] &= 2\theta^2 - (\theta)^2 = \theta^2 = (\mathrm{E}[X])^2 \\
\mathrm{CV}(X) &= \frac{\theta}{\theta} = 1
\end{aligned}$$

Example

An insurance company classifies their policyholders as either high risk, with annual losses that are exponential with mean 100, or low risk, with annual losses that are exponential with mean 10. Let X be last year's loss amount of a randomly selected policyholder. If 30% of policyholders are high risk, what is $\mathrm{Var}(X)$?

$$\begin{aligned}
\mathrm{E}[X \mid \text{High}] &= 100 \quad \mathrm{E}[X \mid \text{Low}] = 10 \\
\mathrm{Var}[X \mid \text{High}] &= 100^2 \quad \mathrm{Var}[X \mid \text{Low}] = 10^2 \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \text{group}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{group}]] \\
&= \left(100^2 \cdot 0.3 + 10^2 \cdot 0.7\right) + (100 - 10)^2(0.3)(0.7) \\
&= 4{,}771
\end{aligned}$$

Memoryless property:

For exponential variables,

$$\begin{aligned}
\mathrm{P}[X - d > x \mid X > d] &= \frac{\mathrm{P}[X > x + d, X > d]}{\mathrm{P}[X > d]} \\
&= \frac{e^{-(x+d)/\theta}}{e^{-d/\theta}} \\
&= e^{-x/\theta} \\
&= S(x)
\end{aligned}$$

i.e., the distribution of $X - d$, given that $X > d$, is the same as the original distribution of X. In particular,

$$\begin{aligned}
\mathrm{E}[X - d \mid X > d] &= e_X(d) = \mathrm{E}[X] = \theta \\
\mathrm{Var}[X - d \mid X > d] &= \mathrm{Var}[X]
\end{aligned}$$

Shifted Exponential Distribution

Suppose that

$$f(x) = \begin{cases} \dfrac{1}{\theta} e^{-(x-\delta)/\theta} & x > \delta \\ 0 & \text{otherwise} \end{cases}$$

X is called a shifted exponential: $X = Y + \delta$, where $Y \sim \exp(\theta)$

$$\begin{aligned} \mathrm{E}[X] &= \theta + \delta \\ \mathrm{Var}[X] &= \theta^2 \end{aligned}$$

Note that the range of X depends on δ

Exercise 1

X is exponential with mean 20. Find $\mathrm{E}[X \mid X > 30]$ and $\mathrm{Var}[X \mid X > 30]$.

Exercise 1

X is exponential with mean 20. Find $\mathrm{E}[X \mid X > 30]$ and $\mathrm{Var}[X \mid X > 30]$.

$$\begin{aligned}
\mathrm{E}[X \mid X > 30] &= \mathrm{E}[X - 30 + 30 \mid X > 30] \\
&= \mathrm{E}[X - 30 \mid X > 30] + 30 \\
&= \mathrm{E}[X] + 30 \\
&= 20 + 30 = \boxed{50} \\
\mathrm{Var}[X \mid X > 30] &= \mathrm{Var}[X - 30 + 30 \mid X > 30] \\
&= \mathrm{Var}[X - 30 \mid X > 30] \\
&= \mathrm{Var}[X] \\
&= 20^2 = \boxed{400}
\end{aligned}$$

Or, $(X \mid X > 30)$ is a shifted exponential with $\theta = 20$ and $\delta = 30$

$$\begin{aligned}
\mathrm{E}[X \mid X > 30] &= \theta + \delta = 20 + 30 = \boxed{50} \\
\mathrm{Var}[X \mid X > 30] &= \theta^2 = \boxed{400}
\end{aligned}$$

Exercise 2

30% of losses are exponential with mean 10, and 70% are from a shifted exponential with mean 25 and variance 400.
Find the mean and variance of a randomly selected loss.

Exercise 2

30% of losses are exponential with mean 10, and 70% are from a shifted exponential with mean 25 and variance 400.
Find the mean and variance of a randomly selected loss.

$$\begin{aligned}
\mathrm{E}[X] &= 0.3 \cdot 10 + 0.7 \cdot 25 = \boxed{20.5} \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \mathrm{Case}]] + \mathrm{Var}[\mathrm{E}[X \mid \mathrm{Case}]] \\
&= 0.3 \cdot 10^2 + 0.7 \cdot 400 + (25 - 10)^2 \cdot 0.3 \cdot 0.7 \\
&= \boxed{357.25}
\end{aligned}$$

Exercise 3

30% of losses are exponential with mean 10, and 70% are from a shifted exponential with mean 25 and variance 400.
Find the probability that a randomly selected loss will exceed 20.

Exercise 3

30% of losses are exponential with mean 10, and 70% are from a shifted exponential with mean 25 and variance 400.
Find the probability that a randomly selected loss will exceed 20.

For the shifted exponential piece, $\theta^2 = 400$ so $\theta = 20$.
$\theta + \delta = 25$, so $\delta = 5$.

$$\begin{aligned}
\mathrm{P}[X > 20] &= 0.3 \cdot \mathrm{P}[\exp(\theta = 10) > 20] + 0.7 \cdot \mathrm{P}[\text{shifted exp} > 20] \\
&= 0.3 \cdot \mathrm{P}[\exp(\theta = 10) > 20] + 0.7 \cdot \mathrm{P}[\exp(\theta = 20) > 15] \\
&= 0.3 \cdot e^{-20/10} + 0.7 e^{-15/20} \\
&= \boxed{0.371}
\end{aligned}$$

A.2 Key Continuous Distributions - Outline

Gamma Density

A Gamma(α, θ) random variable has density

$$
\begin{aligned}
f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)} \quad \text{(table version)} \\
&= \frac{x^{\alpha-1}}{\theta^\alpha \Gamma(\alpha)} e^{-x/\theta}
\end{aligned}
$$

If α is an integer then

$$
\begin{aligned}
\Gamma(\alpha) &= (\alpha - 1)! \\
&= (\alpha - 1) \cdot (\alpha - 2)! \\
\Gamma(\alpha) &= (\alpha - 1) \cdot \Gamma(\alpha - 1) \text{ for any } \alpha > 1
\end{aligned}
$$

Old exams have given you $\Gamma(\alpha)$ when α is not an integer

$$
\begin{aligned}
\text{e.g., } \Gamma(1/2) &= \sqrt{\pi} \\
\Gamma(3/2) &= \left(\frac{3}{2} - 1\right)\sqrt{\pi} = \frac{1}{2}\sqrt{\pi}
\end{aligned}
$$

Gamma CDF

Suppose α is an integer.
(This case is also called the Erlang distribution.)
If $X_1, \ldots, X_\alpha$ are iid exponential(θ) random variables, then
$X_1 + \cdots + X_\alpha \sim$ Gamma(α, θ).
α is the *shape* parameter
θ is the *scale* parameter.

The exam tables don't really include the CDF. They say that

$$F(x) = \Gamma\left(\alpha; \frac{x}{\theta}\right)$$

If you look up the definition, that says that

$$F(x) = \int_0^x f(t)\, dt$$

which we already knew.

Gamma CDF

If α is an integer, the CDF is nice. Memorize it at least up to $\alpha = 2$.

$$\alpha = 1 : F(x) = 1 - e^{-x/\theta}$$

$$\alpha = 2 : F(x) = 1 - e^{-x/\theta} - \frac{x}{\theta}e^{-x/\theta}$$

$$\alpha = 3 : F(x) = 1 - e^{-x/\theta} - \frac{x}{\theta}e^{-x/\theta} - \frac{(x/\theta)^2}{2}e^{-x/\theta}$$

$$\text{General } \alpha : F(x) = 1 - \sum_{i=0}^{\alpha-1} \mathrm{P}[\text{Poisson}(x/\theta) = i]$$

The survival function is:

$$\alpha = 1 : S(x) = e^{-x/\theta}$$

$$\alpha = 2 : S(x) = e^{-x/\theta} + \frac{x}{\theta}e^{-x/\theta}$$

$$\alpha = 3 : S(x) = e^{-x/\theta} + \frac{x}{\theta}e^{-x/\theta} + \frac{(x/\theta)^2}{2}e^{-x/\theta}$$

Mean and Variance

When α is an integer, a Gamma(α, θ) random variable is the sum of α independent exponential(θ) variables.

So $\mathrm{E}[X] = \alpha \cdot \theta$ and $\mathrm{Var}[X] = \alpha \cdot \theta^2$.

When α is not an integer, those two formulas still hold.

Example

The number of annual losses N satisfies $\mathrm{P}[N=0]=0.2$, $\mathrm{P}[N=1]=0.5$ and $\mathrm{P}[N=2]=0.3$. Loss amounts are independent exponentials with mean 10. What are the mean and variance of the annual loss amounts?
Let S denote the annual loss amount.

$$\begin{aligned}
\mathrm{E}[S] &= \mathrm{E}[S \mid N=0]\cdot \mathrm{P}[N=0] + \mathrm{E}[S \mid N=1]\cdot \mathrm{P}[N=1] \\
&\quad + \mathrm{E}[S \mid N=2]\cdot \mathrm{P}[N=2] \\
&= 0\cdot 0.2 + 10\cdot 0.5 + 20\cdot 0.3 = 11 \\
\mathrm{E}[S] &= \mathrm{E}[\mathrm{E}[S \mid N]] = \mathrm{E}[10N] = 10(0\cdot 0.2 + 1\cdot 0.5 + 2\cdot 0.3) = 11 \\
\mathrm{Var}[S] &= \mathrm{E}[\mathrm{Var}[S \mid N]] + \mathrm{Var}[\mathrm{E}[S \mid N]] \\
&= \mathrm{E}[100N] + (0^2\cdot 0.2 + 10^2\cdot 0.5 + 20^2\cdot 0.3 - 11^2) \\
&= 110 + 49 = 159
\end{aligned}$$

Gamma CDF Revisited

The exam tables list the Gamma CDF as

$$F(x) = \Gamma\left(\alpha; \frac{x}{\theta}\right)$$

If α is an integer, we saw

$$\Gamma\left(\alpha; \frac{x}{\theta}\right) = 1 - \sum_{i=0}^{\alpha-1} e^{-x/\theta}\cdot \frac{(x/\theta)^i}{i!}$$

$\Gamma(\alpha; y) =$ *incomplete Gamma function.*
To evaluate it (only need to do for $\alpha = 1$ or 2),
$\Gamma(\alpha; y) =$ cdf of a Gamma with $\theta = 1$ evaluated at y.

Some other exam table formulas also require the incomplete Gamma function.

Examples

Suppose that X is an inverse Gamma random variable with parameters $\alpha = 3$ and $\theta = 50$. What is $\mathrm{P}[X < 100]$?

From the table, $F(x) = 1 - \Gamma\left(\alpha; \dfrac{\theta}{x}\right)$.

Note the reversals from the Gamma cdf; in particular, we now have θ/x instead of x/θ.

$$\begin{aligned}
F(100) = 1 - \Gamma\left(3; \frac{50}{100}\right) &= 1 - \Gamma\left(3; \frac{1}{2}\right) \\
&= 1 - \left[1 - \sum_{i=0}^{2} e^{-1/2} \cdot \frac{(1/2)^i}{i!}\right] \\
&= e^{-1/2} + \frac{1}{2}e^{-1/2} + \frac{(1/2)^2}{2!}e^{-1/2} \\
&= \frac{13}{8}e^{-1/2}
\end{aligned}$$

Exercise 1

X is a Gamma random variable with mean 15 and variance 75. Find $\mathrm{P}[X \leq 10]$.

Exercise 1

X is a Gamma random variable with mean 15 and variance 75. Find $\mathrm{P}[X \leq 10]$.

$$
\begin{aligned}
\mathrm{E}[X] &= \alpha\theta = 15 \\
\mathrm{Var}[X] &= \alpha\theta^2 = 75 \\
\theta &= \frac{75}{15} = 5 \\
\alpha &= 3 \\
\mathrm{P}[X \leq 10] &= 1 - e^{-10/\theta} - \left(\frac{10}{\theta}\right) e^{-10/\theta} - \dots \\
&= 1 - e^{-10/5} - \left(\frac{10}{5}\right) e^{-2} - \frac{(10/5)^2}{2!} e^{-2} \\
&= 1 - e^{-2}(1 + 2 + 2) \\
&= \boxed{0.3233}
\end{aligned}
$$

Exercise 2

The number of annual losses N satisfies $\mathrm{P}[N = 0] = 0.2$, $\mathrm{P}[N = 1] = 0.5$ and $\mathrm{P}[N = 2] = 0.3$. Loss amounts are independent exponentials with mean 10. What is the probability that annual losses will exceed 15?

Exercise 2

The number of annual losses N satisfies $\mathrm{P}[N = 0] = 0.2$, $\mathrm{P}[N = 1] = 0.5$ and $\mathrm{P}[N = 2] = 0.3$. Loss amounts are independent exponentials with mean 10. What is the probability that annual losses will exceed 15?

Let S denote the annual loss amount.

$$\begin{aligned}
\mathrm{P}[S > 15] &= \mathrm{P}[N = 0] \cdot \mathrm{P}[S > 15 \mid N = 0] \\
&\quad + \mathrm{P}[N = 1] \cdot \mathrm{P}[S > 15 \mid N = 1] \\
&\quad + \mathrm{P}[N = 2] \cdot \mathrm{P}[S > 15 \mid N = 2] \\
&= 0.2 \cdot 0 + 0.5 \cdot e^{-15/10} + 0.3 \cdot \left(1 + \frac{15}{10}\right) e^{-15/10} \\
&= \boxed{0.279}
\end{aligned}$$

A.2 Key Continuous Distributions - Outline

Pareto

"Pareto" by itself refers to the 2-parameter Pareto (A.2.3.1 on the formula sheet).

$$
\begin{aligned}
\mathrm{P}[X > x] &= \left(\frac{\theta}{x+\theta}\right)^{\alpha} \quad x > 0 \\
\mathrm{E}[X] &= \frac{\theta}{\alpha - 1} \quad \text{if } \alpha > 1 \\
\mathrm{E}[X^2] &= \frac{2\theta^2}{(\alpha-1)(\alpha-2)} \quad \text{if } \alpha > 2 \\
\mathrm{Var}[X] &= \left(\frac{\theta}{\alpha-1}\right)^2 \cdot \frac{\alpha}{\alpha-2} \\
&= (\mathrm{E}[X])^2 \cdot \frac{\alpha}{\alpha-2}
\end{aligned}
$$

Note that $\alpha/(\alpha-2) \to 1$ as $\alpha \to \infty$ and $\alpha/(\alpha-2) \to \infty$ as $\alpha \downarrow 2$

Pareto

The Pareto isn't quite memoryless:

$$
\begin{aligned}
\mathrm{P}[X - d > x \mid X > d] &= \frac{\mathrm{P}[X > x + d]}{\mathrm{P}[X > d]} \\
&= \frac{\left(\dfrac{\theta}{x+d+\theta}\right)^{\alpha}}{\left(\dfrac{\theta}{d+\theta}\right)^{\alpha}} \\
&= \left(\frac{d+\theta}{x+d+\theta}\right)^{\alpha}
\end{aligned}
$$

i.e., $(X - d \mid X > d)$ is again a Pareto with the same α and a new $\theta' = \theta + d$.

So $\mathrm{E}[X - d \mid X > d] = \dfrac{\theta + d}{\alpha - 1}$

and $\mathrm{Var}[X - d \mid X > d] = \left(\dfrac{\theta + d}{\alpha - 1}\right)^2 \cdot \dfrac{\alpha}{\alpha - 2}$

Single Parameter Pareto

The "single parameter" Pareto has range $X > \theta$ and density

$$f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}} \quad x > \theta$$

A single parameter Pareto is a Pareto distribution shifted by θ.

$$\begin{aligned} \mathrm{E}[X] &= \frac{\theta}{\alpha-1} + \theta = \frac{\theta + (\alpha-1)\theta}{\alpha-1} \\ &= \frac{\alpha\theta}{\alpha-1} \quad \text{(in tables)} \\ \mathrm{Var}[X] &= \mathrm{Var}[\text{Pareto}] \\ &= \left(\frac{\theta}{\alpha-1}\right)^2 \cdot \frac{\alpha}{\alpha-2} \end{aligned}$$

It is unlikely that memorizing the variance will be useful.

Exercise 1

X is Pareto distributed with $\mathrm{E}[X] = 5$ and $\mathrm{E}[X^2] = 100$.
Find $\mathrm{P}[X \leq 20 \mid X > 10]$.

Exercise 1

X is Pareto distributed with $\mathrm{E}[X] = 5$ and $\mathrm{E}[X^2] = 100$.
Find $\mathrm{P}[X \leq 20 \mid X > 10]$.

$$\begin{aligned}
\mathrm{E}[X] &= \frac{\theta}{\alpha - 1} = 5 \\
\mathrm{E}[X^2] &= \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = 100 \\
\frac{\mathrm{E}[X^2]}{(\mathrm{E}[X])^2} &= \frac{2(\alpha-1)}{\alpha-2} = 4 \\
4(\alpha-2) &= 2\alpha - 2 \Rightarrow \alpha = 3 \quad \theta = 10 \\
\mathrm{P}[X \leq 20 \mid X > 10] &= \frac{\mathrm{P}[X \leq 20] - \mathrm{P}[X \leq 10]}{\mathrm{P}[X > 10]} \\
&= \frac{\mathrm{P}[X > 10] - \mathrm{P}[X > 20]}{\mathrm{P}[X > 10]} \\
&= \frac{\left(\frac{10}{10+10}\right)^3 - \left(\frac{10}{20+10}\right)^3}{\left(\frac{10}{10+10}\right)^3} = \boxed{\frac{19}{27}}
\end{aligned}$$

Exercise 2

A portfolio consists of 30 liability risks and 20 property risks, all with identical claim count distributions. Loss sizes for liability risks have a Pareto distribution with $\alpha = 4$ and $\theta = 30$. Loss sizes for property risks have a Pareto distribution with $\alpha = 3$ and $\theta = 50$. Find the variance of the loss size for this portfolio for a single claim.

Exercise 2

A portfolio consists of 30 liability risks and 20 property risks, all with identical claim count distributions. Loss sizes for liability risks have a Pareto distribution with $\alpha = 4$ and $\theta = 30$. Loss sizes for property risks have a Pareto distribution with $\alpha = 3$ and $\theta = 50$. Find the variance of the loss size for this portfolio for a single claim.

$$\mathrm{Var}[X] = \mathrm{E}[\mathrm{Var}(X \mid \text{Case})] + \mathrm{Var}[\mathrm{E}(X \mid \text{Case})]$$

$$\mathrm{E}[X \mid \text{liability}] = \frac{30}{4-1} = 10 \quad \mathrm{E}[X \mid \text{property}] = \frac{50}{3-1} = 25$$

$$\mathrm{Var}[X \mid \text{liability}] = 10^2 \cdot \frac{4}{4-2} = 200$$

$$\mathrm{Var}[X \mid \text{property}] = 25^2 \cdot \frac{3}{3-2} = 1{,}875$$

$$\mathrm{Var}[X] = \left[\frac{30}{50} \cdot 200 + \frac{20}{50} \cdot 1{,}875\right] + (25-10)^2 \cdot 0.6 \cdot 0.4 = \boxed{924}$$

A.2 Key Continuous Distributions - Outline

Normal

A standard normal $Z \sim N(0,1)$ has cdf $\Phi(z)$ and density

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{(on tables)}$$

$$\int_0^\infty \phi(z)\,dz = 1 - \Phi(0) = \frac{1}{2}$$

$$\int_0^\infty e^{-z^2/2}\,dz = \frac{\sqrt{2\pi}}{2}$$

$\Phi(z)$ is given on tables for $z > 0$.
For $z < 0$, use $\Phi(z) = 1 - \Phi(-z)$.
If $t < 0.5$, then $\Phi^{-1}(t) = -\Phi^{-1}(1-t)$.

When looking up $\Phi(z)$ or $\Phi^{-1}(z)$, round to the nearest value on the table. *Don't interpolate!*
E.g., $\Phi(-1.234) = 1 - \Phi(1.234) = 1 - \Phi(1.23) = 0.1093$.

Sums and Transformations

If $X \sim N(\mu, \sigma^2)$ then $X = \sigma Z + \mu$ where $Z \sim N(0,1)$

$$\mathrm{P}[X \le x] = \mathrm{P}\left[\frac{X-\mu}{\sigma} \le \frac{x-\mu}{\sigma}\right] = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Linear combinations (e.g., sums and weighted averages) of independent (or just bivariate) normals are normal.

Example

If $X \sim N(10, 50)$, and $Y \sim N(20, 60)$ are independent, find $\mathrm{P}[X + Y > 40]$.

$$\mathrm{E}[X+Y] = \mathrm{E}[X] + \mathrm{E}[Y] = 30$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] = 110$$

$$\begin{aligned}\mathrm{P}[X+Y > 40] &= 1 - \Phi\left(\frac{40-30}{\sqrt{110}}\right) \\ &\approx 1 - \Phi(0.95) \\ &= \boxed{0.1711}\end{aligned}$$

Mixture

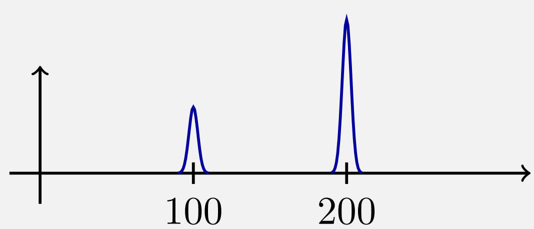
Losses from low risk individuals are $N(100, 1)$, while losses from high risk individuals are $N(200, 1)$. If 30% of losses are from low risk individuals, what is the probability that a randomly selected loss exceeds 200?
What is the mean and variance of a randomly selected loss?

$$\begin{aligned}
\mathrm{P}[X > 200] &= 0.3 \cdot \mathrm{P}[N(100, 1) > 200] + 0.7 \cdot \mathrm{P}[N(200, 1) > 200] \\
&\approx 0.3 \cdot 0 + 0.7 \cdot 0.5 = 0.35 \\
\mathrm{E}[X] &= 0.3 \cdot 100 + 0.7 \cdot 200 = 170 \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{Case}]] \\
&= 0.3 \cdot 1 + 0.7 \cdot 1 + (200 - 100)^2 \cdot 0.3 \cdot 0.7 = 2{,}101
\end{aligned}$$

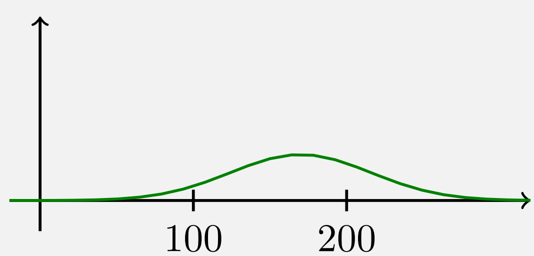
Note: A normal has a single mode at the mean. But instead of having a single mode at 170, our distribution is bimodal.
Bottom line: *Mixtures of normals are not normal.*

Mixture Example Continued

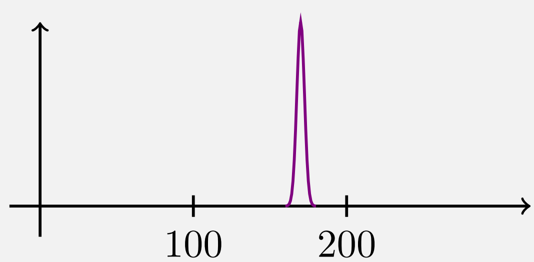
That was a two-point mixture:
$X \sim N(100, 1)$ 30% of the time,
$X \sim N(200, 1)$ 70% of the time.
The graph of $f(x)$ is



$\mathrm{E}[X] = 170, \mathrm{Var}[X] = 2{,}101$.
If $Y \sim N(170, 2{,}101)$, then $f(y)$ looks like



If
$Z_1 \sim N(100, 1), Z_2 \sim N(200, 1)$
are independent.
$W = 0.3Z_1 + 0.7Z_2$
W is a sum so is normal
$\mathrm{E}[W] = 170, \mathrm{Var}[W] = 0.58$
$f(w)$ looks like



Lognormal

Suppose $Y \sim N(\mu, \sigma^2)$ and $X = e^Y$. Then X is a lognormal. The tables use the same μ and σ^2 as the parameters of X as well.

For moments of X we use the tables. If $\mu = 2$ and $\sigma = 1.4$ then

$$\mathrm{E}[X] = e^{\mu+\sigma^2/2} = e^{2+1.4^2/2} = 19.7$$

$$\mathrm{E}[X^2] = e^{2\mu+2^2\sigma^2/2} = e^{4+2\cdot 1.4^2} = 2{,}752$$

Note: μ and σ^2 are the mean and variance of the underlying normal Y and not of the lognormal X.

For probabilities, we take logs and work with the normal.

$$\begin{aligned}\mathrm{P}[X > 19.7] &= \mathrm{P}[Y > \ln(19.7)] \\ &= 1 - \mathrm{P}\left[\frac{Y-2}{1.4} \leq \frac{\ln(19.7)-2}{1.4}\right] \\ &= 1 - \Phi(0.7) = 0.242\end{aligned}$$

Exercise 1

X is lognormal with mean 10 and variance 200. Find $\mathrm{P}[X > 10]$.

Exercise 1

X is lognormal with mean 10 and variance 200. Find $\mathrm{P}[X > 10]$.

$$\begin{aligned}
\mathrm{E}[X] &= 10 = e^{\mu+\sigma^2/2} \\
\mathrm{E}[X^2] &= 10^2 + 200 = e^{2\mu+2\sigma^2} \\
2\mu + \sigma^2 &= 2\ln(10) \\
2\mu + 2\sigma^2 &= \ln(300) \\
\sigma^2 &= \ln(300) - 2\ln(10) = 1.099 \\
\mu &= 1.753 \\
\mathrm{P}[X > 10] &= \mathrm{P}[\ln(X) > \ln(10)] \\
&= \mathrm{P}\left[\frac{\ln(X) - \mu}{\sigma} > \frac{\ln(10) - 1.753}{\sqrt{1.099}}\right] \\
&= 1 - \Phi(0.52) = \Phi(-0.52) \\
&= \boxed{0.3015}
\end{aligned}$$

Exercise 2

Suppose that X is normal with $\mathrm{P}[X > 5] = 0.5$ and $\mathrm{P}[X > 8] = 0.05$. Find $\mathrm{E}[e^{2X}]$.

Exercise 2

Suppose that X is normal with $\mathrm{P}[X > 5] = 0.5$ and $\mathrm{P}[X > 8] = 0.05$. Find $\mathrm{E}[e^{2X}]$.

$$\begin{aligned}
\mu &= 5 \\
\frac{8-5}{\sigma} &= 1.645 \\
\sigma &= 1.8237 \\
e^X &\sim \mathrm{LN}(\mu = 5, \sigma^2 = 3.3259) \\
\mathrm{E}[e^{2X}] &= \mathrm{E}[(e^X)^2] \\
&= e^{2\mu + 2\sigma^2} \\
&= \boxed{17{,}052{,}771}
\end{aligned}$$

A.2 Key Continuous Distributions - Outline

Continuous Uniform Variables

Suppose that X is uniform on (a, b).

Then all points in the interval are "equally likely" so for $a < x < b$,

$$\begin{aligned} f(x) &= \frac{1}{\text{length of interval}} = \frac{1}{b-a} \\ F(x) &= \frac{\text{amount of interval} \leq x}{\text{length of interval}} = \frac{x-a}{b-a} \\ \mathrm{E}[X] &= \frac{a+b}{2} \\ \mathrm{Var}[X] &= \frac{(b-a)^2}{12} \end{aligned}$$

Example

For low risk insureds, losses are uniform on $(0, 10)$. For high risk insureds, losses are uniform on $(4, 20)$. If 30% of insureds are low risk, what is the survival function for a randomly selected loss amount?

$$\begin{aligned} \mathrm{P}[X > x] &= 0.3 \cdot \mathrm{P}[X > x \mid \text{low}] + 0.7 \cdot \mathrm{P}[X > x \mid \text{high}] \\ &= \begin{cases} 0.3 \cdot \dfrac{10-x}{10} + 0.7 \cdot 1 & 0 < x < 4 \\ 0.3 \cdot \dfrac{10-x}{10} + 0.7 \cdot \dfrac{20-x}{16} & 4 < x < 10 \\ 0.3 \cdot 0 + 0.7 \cdot \dfrac{20-x}{16} & 10 < x < 20 \end{cases} \end{aligned}$$

Formula comparison

For $\alpha \geq 1$ an integer,

	Exponential	Gamma	Pareto	Uniform(a, b)
$f(x)$	$\frac{1}{\theta}e^{-x/\theta}$	$\frac{x^{\alpha-1}}{\theta^\alpha(\alpha-1)!}e^{-x/\theta}$	$\frac{\alpha\theta^\alpha}{(\theta+x)^{\alpha+1}}$	$\frac{1}{b-a}$
$F(x)$	$1-e^{-x/\theta}$	$1-\sum_{i=0}^{\alpha-1}\frac{(x/\theta)^i}{i!}e^{-x/\theta}$	$1-\left(\frac{\theta}{x+\theta}\right)^\alpha$	$\frac{x-a}{b-a}$
$S(x)$	$e^{-x/\theta}$	$\sum_{i=0}^{\alpha-1}\frac{(x/\theta)^i}{i!}e^{-x/\theta}$	$\left(\frac{\theta}{x+\theta}\right)^\alpha$	$\frac{b-x}{b-a}$
$\mathrm{E}[X]$	θ	$\alpha\theta$	$\frac{\theta}{\alpha-1}$	$\frac{a+b}{2}$
$\mathrm{Var}[X]$	θ^2	$\alpha\theta^2$	$\frac{\theta^2\alpha}{(\alpha-1)^2(\alpha-2)}$	$\frac{(b-a)^2}{12}$
$e_X(d)$	θ	messy	$\frac{\theta+d}{\alpha-1}$	$\frac{b-d}{2}$

where $e_X(d) = \mathrm{E}[X-d \mid X>d]$

Exercise 1

Low risk individuals have losses that are uniformly distributed on $(0, 50)$, while high risk individuals have losses that are uniform on $(0, 100)$. The expected loss size of a randomly chosen individual is 35. What is the variance of the loss amount of a randomly chosen individual?

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Let $p = \mathrm{P}[\text{low}]$. Then

$$\begin{aligned}
\mathrm{E}[X] &= p \cdot \mathrm{E}[X \mid \text{low}] + (1-p) \cdot \mathrm{E}[X \mid \text{high}] \\
35 &= p \cdot 25 + (1-p) \cdot 50 \\
p &= 0.6 \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{Case}]] \\
&= 0.6 \cdot \frac{50^2}{12} + 0.4 \cdot \frac{100^2}{12} + (25-50)^2 \cdot 0.6 \cdot 0.4 \\
&= \boxed{608.33}
\end{aligned}$$

Exercise 2

Suppose that X is an exponential random variable with mean μ, where μ is a Pareto random variable with parameters $\alpha = 3$ and $\theta = 100$. Find the variance of X.

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$$\begin{aligned}
\operatorname{Var}[X] &= \operatorname{E}[\operatorname{Var}(X \mid \mu)] + \operatorname{Var}[\operatorname{E}(X \mid \mu)] \\
&= \operatorname{E}[\mu^2] + \operatorname{Var}[\mu] \\
&= \frac{2 \cdot 100^2}{(3-1)(3-2)} + \left(\frac{100}{3-1}\right)^2 \frac{3}{3-2} \\
&= \boxed{17{,}500}
\end{aligned}$$