The Infinite Actuary Exam 4/C Online Seminar
A.2. Key Continuous Distributions Solutions
Last updated December 3, 2015

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1. First, let's find the fourth central moment.

$$\begin{aligned}
\mathrm{E}\left[(X-\mu)^4\right] &= \mathrm{E}\left[X^4 - 4X^3\mu + 6X^2\mu^2 - 4X\mu^3 + \mu^4\right] \\
&= \mathrm{E}[X^4] - 4\mu\mathrm{E}[X^3] + 6\mu^2\mathrm{E}[X^2] - 4\mu^3\mathrm{E}[X] + \mu^4 \\
&= 10^4 \cdot 4! - 40 \cdot 3! \cdot 10^3 + 600 \cdot 2! \cdot 10^2 - 4{,}000 \cdot 10 + 10^4 = 9 \cdot 10^4
\end{aligned}$$

The variance is 10^2, so the kurtosis is $9 \cdot 10^4/(10^2)^{4/2} = \boxed{9}$

2.

$$\begin{aligned}
\mathrm{E}[X] &= 0.7 \cdot 10 + 0.3 \cdot (10+\delta) \\
13 &= 7 + 3 + 0.3\delta \Rightarrow \delta = 10 \\
\mathrm{P}[\text{Shifted exp} > t] &= \mathrm{P}[\exp(10) + \delta > t] \\
&= \mathrm{P}[\exp(10) > t - \delta] = e^{-(t-\delta)/10} \\
\mathrm{P}[X > 15] &= 0.7e^{-15/10} + 0.3e^{-(15-10)/10} = \boxed{0.338}
\end{aligned}$$

3.

$$\begin{aligned}
\mathrm{E}[X] &= p \cdot 10 + (1-p) \cdot (10+\delta) \\
13 &= 10p + 10 - 10p + (1-p)\delta = 10 + (1-p)\delta \\
3 &= (1-p)\delta \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{Case}]] \\
136 &= p \cdot 100 + (1-p) \cdot 100 + \delta^2 \cdot p \cdot (1-p) \\
36 &= \delta p \cdot \delta(1-p) = \delta p \cdot 3 \\
12 &= \delta p \\
3 &= (1-p)\delta = \delta - \delta p = \delta - 12 \\
\delta &= 15 \Rightarrow p = \boxed{0.8}
\end{aligned}$$

Or we could have used the 2nd raw moment:

$$\begin{aligned}
\mathrm{E}[X^2] &= p \cdot 2 \cdot 10^2 + (1-p) \cdot [(10+\delta)^2 + 10^2] \\
136 + 13^2 &= p \cdot 200 + (1-p) \cdot [200 + 20\delta + \delta^2] \\
305 &= 200 + 20(1-p)\delta + (1-p)\delta \cdot \delta \\
105 &= 20 \cdot 3 + 3 \cdot \delta \\
\delta &= 15 \Rightarrow p = \boxed{0.8}
\end{aligned}$$

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4. Note the differences between this and the previous problem. Here, we have two exponentials, whereas in the other one we had an exponential and a shifted exponential.

$$\begin{aligned}
\mathrm{E}[X] &= p \cdot 10 + (1-p) \cdot (10+\delta) \\
12 &= 10 + (1-p)\delta
\end{aligned}$$

$$
\begin{aligned}
2 &= (1-p)\delta \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{Case}]] \\
156 &= p \cdot 100 + (1-p) \cdot (10+\delta)^2 + \delta^2 \cdot p \cdot (1-p) \\
156 &= 100p + 100(1-p) + 20\delta(1-p) + \delta^2(1-p) + \delta p \cdot \delta(1-p) \\
156 &= 100 + 20 \cdot 2 + \delta \cdot 2 + \delta p \cdot 2 \\
16 &= 2\delta + 2(\delta - 2) = 4\delta - 4 \\
\delta &= 5 \Rightarrow p = \boxed{0.6}
\end{aligned}
$$

Or using raw moments:

$$
\begin{aligned}
\mathrm{E}[X^2] &= p \cdot 2 \cdot 10^2 + (1-p) \cdot 2 \cdot (10+\delta)^2 \\
156 + 12^2 &= 200p + (1-p)(200 + 40\delta + 2\delta^2) \\
300 &= 200 + 40(1-p)\delta + 2(1-p)\delta \cdot \delta \\
100 &= 40 \cdot 2 + 2 \cdot 2 \cdot \delta \\
\delta &= 5 \Rightarrow p = \boxed{0.6}
\end{aligned}
$$

5.

$$
\begin{aligned}
0.5 = \mathrm{P}[X \le t \mid X > 100] &= \frac{\mathrm{P}[100 < X \le t]}{\mathrm{P}[X > 100]} \\
0.5 &= \frac{e^{-100/50} - e^{-t/50}}{e^{-100/50}} \\
e^{-t/50} &= 0.5e^{-100/50} \\
\frac{-t}{50} &= \ln(0.5) - 2 \\
t &= \boxed{134.66}
\end{aligned}
$$

Alternatively, by the memoryless property, t is 100 + median of X, so $t = 100 + -50\ln(0.5) = 134.66$

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6. Let X and Y be our two losses.

$$
\begin{aligned}
0.5 &= 1 - e^{-50/\theta} \\
\theta &= \frac{-50}{\ln(0.5)} = 72.135 \\
\mathrm{P}[X > 100 \cup Y > 100] &= 1 - \mathrm{P}[X \le 100, Y \le 100] \\
&= 1 - \left(1 - e^{-100/\theta}\right)^2 \\
&= 1 - \frac{9}{16} = \frac{7}{16} = \boxed{0.4375}
\end{aligned}
$$

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7. Again let X and Y be the two losses. From before, $\theta = 72.135$, and

$$
\mathrm{P}[X > 100, Y > 100] = \left(e^{-100/\theta}\right)^2 = \frac{1}{16} = \boxed{0.0625}
$$

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8. $X + Y \sim \text{Gamma}(\alpha = 2, \theta = 72.135)$, and

$$
\mathrm{P}[X + Y > 100] = e^{-100/\theta} + \frac{100}{\theta}e^{-100/\theta} = \boxed{0.5966}
$$

9.
$$\begin{aligned} f(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha} e^{-x/\theta} = cx^{\alpha-1}e^{-x/\theta} \\ f'(x) &= c(\alpha-1)x^{\alpha-2}e^{-x/\theta} - c\cdot\frac{1}{\theta}x^{\alpha-1}e^{-x/\theta} \\ 0 &= (\alpha-1) - \frac{x}{\theta} \\ x &= \boxed{(\alpha-1)\theta} \end{aligned}$$

Remark: Our critical value is a maximum since $f(x) \to 0$ as $x \to 0$ or $x \to \infty$

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10. From the tables, $\mathrm{E}[X^k] = \dfrac{\alpha\theta^k}{\alpha - k}$ so $8 = \dfrac{4\alpha}{\alpha-1}$ and thus $\alpha = 2$.

We want $\mathrm{P}[X > 6] = S(6) = 1 - F(6) = 1 - \left(1 - \left(\frac{4}{6}\right)^2\right) = \left(\frac{2}{3}\right)^2 = \boxed{\frac{4}{9}}$

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11. The raw moments are in the tables, giving us

$$\begin{aligned} \mu &= \frac{\theta}{\alpha-1} = 333.3 \\ \mu_2' &= \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = 333{,}333.3 \\ \mu_3' &= \frac{6\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} = 10^9 \end{aligned}$$

Therefore $\sigma^3 = \mathrm{Var}(X)^{3/2} = \left(E(X^2) - \mu^2\right)^{3/2} = (222{,}222)^{1.5} = 104{,}756{,}560$ and

$$\begin{aligned} \mu_3 &= \mathrm{E}\left[(X-\mu)^3\right] = \mathrm{E}[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3] \\ &= \mathrm{E}[X^3] - 3\mu\mathrm{E}[X^2] + 3\mu^2\mu - \mu^3 \\ &= 740{,}740{,}741 \end{aligned}$$

and so the skewness is $740.740/104.757 = \boxed{7.07}$

Later on, we will see that the answer does't depend on θ, so you can set $\theta = 1$ to make the numbers easier to work with.

12.
$$\begin{aligned} F(x) &= 0.3\left[1 - \left(\frac{10}{x+10}\right)^3\right] + 0.7\left[1 - \left(\frac{10}{x+10}\right)^6\right] \\ 0.5 &= 0.3 - 0.3u + 0.7 - 0.7u^2 \quad \text{where } u = \left(\frac{10}{x+10}\right)^3 \\ 0.7u^2 + 0.3u - 0.5 &= 0 \\ u &= \left(\frac{10}{x+10}\right)^3 = 0.6576 \\ x &= \boxed{1.5} \end{aligned}$$

13. For a Pareto distribution, the distribution of $(X-d \mid X>d)$ is a Pareto with the same α and with an updated $\theta' = \theta + d$. So we get

$$\frac{\theta+100}{2-1} = \frac{5}{3}\frac{\theta+50}{2-1}$$
$$\theta = 25$$

and then $\mathrm{E}[X-150 \mid X>150] = (25+150)/(2-1) = \boxed{175}$

14.
$$\begin{aligned} F(200) &= 0.8\left[1-\left(\frac{100}{100+200}\right)^2\right] + 0.2\left[1-\left(\frac{3{,}000}{3{,}000+200}\right)^4\right] \\ &= 0.711 + 0.046 = \boxed{0.757} \end{aligned}$$

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15. Let M denote the height of a randomly selected male, and F a randomly selected female. They are independent, so $M-F$ is normal with mean $\mathrm{E}[M]-\mathrm{E}[F] = 176-163 = 13$, and variance $\mathrm{Var}[M] + (-1)^2\mathrm{Var}[F] = 36+25 = 61$.
$\mathrm{P}[M>F] = \mathrm{P}\left[\frac{M-F-13}{\sqrt{61}} > \frac{0-13}{\sqrt{61}}\right] = 1-\Phi(\frac{-13}{\sqrt{61}} = \Phi(1.66) = \boxed{0.95}$

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16. Let X be a random claim amount.

$$\begin{aligned} \mathrm{P}[X>6{,}000] &= (0.6)\mathrm{P}[N(5{,}000;1{,}000^2)>6{,}000] + (0.4)\mathrm{P}[N(4{,}000;1{,}000^2)>6{,}000] \\ &= 0.6(1-\Phi(1)) + 0.4(1-\Phi(2)) = 0.6(0.1587) + 0.4(0.0228) \\ &= \boxed{0.104} \end{aligned}$$

Note that the resulting mixture is not a normal random variable. For one thing, it is bi-modal, with the density having one local max at 4,000 and a second one at 5,000, while a normal random variable has a single mode.

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17. Here we have a weighted average, which is a type of sum, so

$$\begin{aligned} \mathrm{E}[0.6X+0.4Y] &= 0.6\cdot 5{,}000 + 0.4\cdot 4{,}000 = 4{,}600 \\ \mathrm{Var}[0.6X+0.4Y] &= 0.6^2\mathrm{Var}[X] + 0.4^2\mathrm{Var}[Y] \\ \mathrm{SD}[0.6X+0.4Y] &= 721.11 \\ \mathrm{P}[0.6X+0.4Y>6{,}000] &= 1-\Phi\left(\frac{6{,}000-4{,}600}{721.11}\right) = 1-\Phi(1.94) \\ &= \boxed{0.0262} \end{aligned}$$

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18. Neither A nor B is normally distributed because there is a positive probability of no claim being made. To use what we know about normal distributions, we have to condition on a payment being made for both A and B.

I will use $A > 0$ and $B > 0$ to denote the case when A and B have claims (that is technically wrong as there is a very small probability of the normal variables being negative, but it simplifies the notation).

$$\begin{aligned}
\mathrm{P}[B > A] &= \mathrm{P}[B > A, A > 0] + \mathrm{P}[B > A, A = 0] \\
&= \mathrm{P}[A > 0] \cdot \mathrm{P}[B > A \mid A > 0] + \mathrm{P}[A = 0] \cdot \mathrm{P}[B > A \mid A = 0] \\
&= 0.4 \cdot \mathrm{P}[B > 0] \cdot \mathrm{P}[B > A \mid A, B > 0] + 0.6 \cdot \mathrm{P}[B > 0] \\
&= (0.4)(0.3)\mathrm{P}[B > A \mid A, B > 0] + (0.6)(0.3)
\end{aligned}$$

And when A and B both have claims we have $B - A$ is a normal random variable with parameters

$$\begin{aligned}
\mathrm{E}[B - A \mid A, B > 0] &= \mathrm{E}[B \mid B > 0] - \mathrm{E}[A \mid A > 0] = 9{,}000 - 10{,}000 = -1{,}000 \\
\mathrm{Var}[B - A \mid A, B > 0] &= \mathrm{Var}[B \mid B > 0] + (-1)^2 \mathrm{Var}[A \mid A > 0] \\
&= 2{,}000^2 + 2{,}000^2 = 8{,}000{,}000 \\
\mathrm{SD}[B - A \mid A, B > 0] &= 2{,}828 \\
\mathrm{P}[B - A > 0 \mid A, B > 0] &= \mathrm{P}\left[\frac{B - A - (-1{,}000)}{2{,}828} > \frac{0 - (-1{,}000)}{2{,}828} \mid A, B > 0\right] \\
&= 1 - \Phi\left(\frac{1{,}000}{2{,}828}\right) \\
&= 1 - \Phi(0.35) = 1 - 0.64 = 0.36 \\
\mathrm{P}[B > A] &= 0.4 \cdot 0.3 \cdot 0.36 + 0.6 \cdot 0.3 = \boxed{0.223}
\end{aligned}$$

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19. $1 - 6.68\% = 0.9332$, and looking that up on the normal table that is a z-value of 1.5. So 102 is 1.5 standard deviations above the mean. $1 - 0.0228 = 0.9772$ which has a z-value of 2, so 105 is 2 standard deviations above the mean. That means that

$$\begin{aligned}
105 - 102 &= 0.5\mathrm{SD}[X] \\
\mathrm{SD}[X] &= 6 \\
\mathrm{E}[X] &= 105 - 2 \cdot 6 = 93 \\
\mathrm{P}[X > 95] &= \mathrm{P}\left[\frac{X - \mathrm{E}[X]}{\mathrm{SD}[X]} > \frac{95 - 93}{6}\right] \\
&= 1 - \Phi(0.33) = 1 - 0.63 = \boxed{0.37}
\end{aligned}$$

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20. $\mathrm{Kurtosis}(X) = \dfrac{\mathrm{E}\left[(X - \mu)^4\right]}{\sigma^4}$ and so since the kurtosis of a normal is 3, $\mathrm{E}[Z^4] = 3$ if Z is a standard normal.

$$\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[\mathrm{E}[Y \mid \mathrm{Case}]] \\
&= \mathrm{P}[Y = X_1] \cdot \mathrm{E}[X_1] + \mathrm{P}[Y = X_2] \cdot \mathrm{E}[X_2] = 0.9 \cdot 0 + 0.1 \cdot 0 = 0 \\
\mathrm{Var}[Y] &= \mathrm{E}[\mathrm{Var}[Y \mid \mathrm{Case}]] + \mathrm{Var}[\mathrm{E}[Y \mid \mathrm{Case}]] \\
&= 0.9 \cdot \mathrm{Var}[X_1] + 0.1 \cdot \mathrm{Var}[X_2] + 0.9 \cdot 0.1 \cdot (0 - 0)^2 \\
\mathrm{Var}[Y] &= 3.4 \\
\text{Or: } \mathrm{E}[Y^2] &= \mathrm{E}[\mathrm{E}[Y^2 \mid \mathrm{Case}]] \\
&= \mathrm{P}[Y = X_1] \cdot \mathrm{E}[X_1^2] + \mathrm{P}[Y = X_2] \cdot \mathrm{E}[X_2^2]
\end{aligned}$$

$$
\begin{aligned}
&= 0.9\cdot(0^2+1^2)+0.1\cdot(0^2+5^2)=3.4\\
\operatorname{Var}[Y] &= 3.4-0^2=3.4\\
\mathrm{E}\left[(Y-\mathrm{E}[Y])^4\right] = \mathrm{E}\left[Y^4\right] &= 0.9\cdot\mathrm{E}\left[X_1^4\right]+0.1\cdot\mathrm{E}\left[X_2^4\right]\\
&= 0.9\cdot 3+0.1\cdot 3\cdot 5^4\\
\mathrm{E}\left[Y^4\right] &= 190.2\\
\operatorname{Kurtosis}(Y) &= \frac{190.2}{3.4^{4/2}} = \boxed{16.45}
\end{aligned}
$$

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21. The median of the underlying normal is μ, so the median of the lognormal is e^μ. Finding the other moments in the tables,

$$
\begin{aligned}
\text{Median} &= e^\mu = 3\\
\mathrm{E}[X] &= e^{\mu+\sigma^2/2}=4\\
e^{\sigma^2/2} &= \frac{4}{3}\\
\mathrm{E}[X^2] &= e^{2\mu+2\sigma^2}=(e^\mu)^2\cdot(e^{\sigma^2/2})^4\\
&= 9\cdot\left(\frac{4}{3}\right)^4=\frac{256}{9}\\
\operatorname{Var}[X] &= \mathrm{E}[X^2]-(\mathrm{E}[X])^2\\
&= \frac{256}{9}-16=\frac{112}{9}=\boxed{12.4}
\end{aligned}
$$

22.

$$
\begin{aligned}
\mathrm{E}[X] &= 3 = e^{\mu+\sigma^2/2}\\
\mathrm{E}[X^2] &= 3^2+2^2=e^{2\mu+2\sigma^2}\\
2\ln(3) &= 2\mu+\sigma^2\\
\ln(13) &= 2\mu+2\sigma^2\\
\sigma^2 &= 0.3677\\
\mu &= 0.915\\
\mathrm{P}[X>4\mid X>1] &= \frac{\mathrm{P}[X>4]}{\mathrm{P}[X>1]}\\
\mathrm{P}[X>4] &= \mathrm{P}[\ln(X)>\ln(4)] = 1-\Phi\left(\frac{\ln(4)-\mu}{\sigma}\right)\\
&= 1-\Phi(0.78)=0.2177\\
\mathrm{P}[X>1] &= \mathrm{P}[\ln(X)>\ln(1)] = 1-\Phi\left(\frac{\ln(1)-\mu}{\sigma}\right)\\
&= \Phi(1.51)=0.9345\\
\mathrm{P}[X>4\mid X>1] &= \frac{0.2177}{0.9345}=\boxed{0.233}
\end{aligned}
$$

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23. Let X denote a 1996 loss amount.

$$
\mathrm{P}\left[X > e^{\mu+\ln(k)+\sigma^2/2}\right] = p
$$

$$
\begin{aligned}
\mathrm{P}\left[X \le e^{\mu+\ln(k)+\sigma^2/2}\right] &= 1-p \\
\mathrm{P}\left[\frac{\ln(X)-\mu}{\sigma} \le \frac{\ln(k)+\sigma^2/2}{\sigma}\right] &= 1-p \\
z_{1-p} = -z_p &= \frac{\ln(k)+\sigma^2/2}{\sigma} \\
\sigma^2 + 2z_p\sigma + 2\ln(k) &= 0 \\
\sigma &= \frac{-2z_p \pm \sqrt{4z_p^2 - 4\cdot 2\ln(k)}}{2} \\
&= \boxed{-z_p \pm \sqrt{z_p^2 - 2\ln(k)}}
\end{aligned}
$$

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24. A standard normal has density $f(x) = e^{-x^2/2}/\sqrt{2\pi}$, and a $N(\mu,\sigma^2)$ has density $\dfrac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/(2\sigma^2)}$, so one way to integrate e^{-x^2} is to compare with a normal CDF.

$$
\begin{aligned}
\mathrm{E}[X] &= \int_0^\infty e^{-2x^2}dx \\
&= \sigma\sqrt{2\pi}\int_0^\infty \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/(2\sigma^2)}dx \quad \text{for } 2 = 1/(2\sigma^2) \text{ and } \sigma = 1/2 \\
&= \frac{1}{2}\sqrt{2\pi}\cdot \mathrm{P}[Z>0] \quad \text{for } Z \sim N(0,1/4) \\
&= \frac{1}{2}\sqrt{2\pi}\cdot\frac{1}{2} = \boxed{\sqrt{\frac{\pi}{8}}}
\end{aligned}
$$

Or: X is a Weibull with $\tau = 2$ and $(1/\theta)^\tau = (1/\theta)^2 = 2$ so $\theta = 1/\sqrt{2}$ and $\mathrm{E}[X] = \dfrac{1}{\sqrt{2}}\Gamma(3/2) = \dfrac{1}{\sqrt{2}}\dfrac{1}{2}\sqrt{\pi} = \sqrt{\dfrac{\pi}{8}}$

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25. Now we have

$$
\begin{aligned}
\mathrm{E}[X] &= \int_0^1 1dx + \int_1^\infty e^{-2x^2}dx \\
&= 1 + \sqrt{\frac{\pi}{2}}\mathrm{P}[Z>1] \\
&= 1 + \sqrt{\frac{\pi}{2}}\mathrm{P}\left[\frac{Z-0}{1/2} > \frac{1-0}{1/2}\right] \\
&= 1 + \sqrt{\frac{\pi}{2}}\left(1-\Phi(2)\right) = \boxed{1.029}
\end{aligned}
$$

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26. $f(x) = ce^{-x^2}$ for $x > 0.5$, where $c = 1/\displaystyle\int_{0.5}^\infty e^{-x^2}dx$ so

$$
\mathrm{P}[X \ge 1] = \frac{\int_1^\infty e^{-x^2}dx}{\int_{0.5}^\infty e^{-x^2}dx}
$$

$$
\begin{aligned}
&= \frac{\mathrm{P}[Z>1]}{\mathrm{P}[Z>0.5]} \text{ where } Z \text{ is a normal with mean 0 and variance } 1/2 \\
&= \frac{1-\Phi\left(\frac{1}{1/\sqrt{2}}\right)}{1-\Phi\left(\frac{0.5}{1/\sqrt{2}}\right)} \\
&= \frac{1-0.9207}{1-0.7611} = \boxed{0.33}
\end{aligned}
$$

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27. For a lognormal, the median is e^{μ} and the mean is $e^{\mu+\sigma^2/2}$ so

$$
\begin{aligned}
e^{\mu} &= 1.06 \\
\mu &= \ln 1.06 = 0.0583 \\
e^{\mu+\sigma^2/2} &= 1.08 \\
\mu+\sigma^2/2 &= \ln 1.06 + \sigma^2/2 = \ln 1.08 \\
\sigma &= \boxed{0.1933}
\end{aligned}
$$

28.

$$
\begin{aligned}
e^{\mu+\sigma^2/2} &= 0.42 \qquad e^{2\mu+\sigma^2} = 0.42^2 \\
e^{2\mu+2\sigma^2} &= 1.65 + 0.42^2 = 1.8264 \\
e^{\sigma^2} &= \frac{1.8264}{0.42^2} \quad \text{by division} \\
\sigma &= 1.5288 \\
\ln 0.42 &= \mu + 1.5288^2/2, \quad \mu = -2.036 \\
\mathrm{P}[X>1] &= \mathrm{P}[\ln X > 0] \\
&= \mathrm{P}\left[\frac{\ln X - \mu}{\sigma} > \frac{2.036}{1.5288}\right] = 1 - \Phi(1.33) \\
&= \boxed{0.0918}
\end{aligned}
$$