The Infinite Actuary Exam 4/C Online Seminar
A.2. Key Continuous Distributions Solutions
Last updated December 3, 2015

1. Calculate the kurtosis for the exponential distribution with mean 10.

A. 1 B. 3 C. 6 D. 9 E. 12

First, let's find the fourth central moment.

$$\begin{aligned}
\mathrm{E}\left[(X-\mu)^4\right] &= \mathrm{E}\left[X^4 - 4X^3\mu + 6X^2\mu^2 - 4X\mu^3 + \mu^4\right] \\
&= \mathrm{E}[X^4] - 4\mu\mathrm{E}[X^3] + 6\mu^2\mathrm{E}[X^2] - 4\mu^3\mathrm{E}[X] + \mu^4 \\
&= 10^4 \cdot 4! - 40 \cdot 3! \cdot 10^3 + 600 \cdot 2! \cdot 10^2 - 4{,}000 \cdot 10 + 10^4 = 9 \cdot 10^4
\end{aligned}$$

The variance is 10^2, so the kurtosis is $9 \cdot 10^4/(10^2)^{4/2} = \boxed{9}$

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2. Losses follow a 2-point mixture, that with probability 0.7 comes from an exponential distribution with mean 10, while with probability 0.3 comes from a distribution with density

$$f(x) = \begin{cases} \dfrac{1}{10} e^{-(x-\delta)/10} & x > \delta \\ 0 & \text{otherwise.} \end{cases}$$

If the mean loss amount is 13, what is the probability of a loss exceeding 15?

A. 0.298 B. 0.308 C. 0.318 D. 0.328 E. 0.338

$$\begin{aligned}
\mathrm{E}[X] &= 0.7 \cdot 10 + 0.3 \cdot (10 + \delta) \\
13 &= 7 + 3 + 0.3\delta \Rightarrow \delta = 10 \\
\mathrm{P}[\text{Shifted exp} > t] &= \mathrm{P}[\exp(10) + \delta > t] \\
&= \mathrm{P}[\exp(10) > t - \delta] = e^{-(t-\delta)/10} \\
\mathrm{P}[X > 15] &= 0.7e^{-15/10} + 0.3e^{-(15-10)/10} = \boxed{0.338}
\end{aligned}$$

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3. Losses are a mixture, and with probability p come from an exponential distribution with mean 10, while with probability $1-p$ they come from a distribution with density

$$f(x) = \begin{cases} \dfrac{1}{10} e^{-(x-\delta)/10} & x > \delta \\ 0 & \text{otherwise.} \end{cases}$$

If the mean loss amount is 13, and the variance of losses is 136, what is p?

A. 0.4 B. 0.5 C. 0.6 D. 0.7 E. 0.8

$$\mathrm{E}[X] = p \cdot 10 + (1-p) \cdot (10 + \delta)$$

$$
\begin{aligned}
13 &= 10p + 10 - 10p + (1-p)\delta = 10 + (1-p)\delta \\
3 &= (1-p)\delta \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{Case}]] \\
136 &= p \cdot 100 + (1-p) \cdot 100 + \delta^2 \cdot p \cdot (1-p) \\
36 &= \delta p \cdot \delta(1-p) = \delta p \cdot 3 \\
12 &= \delta p \\
3 &= (1-p)\delta = \delta - \delta p = \delta - 12 \\
\delta &= 15 \Rightarrow p = \boxed{0.8}
\end{aligned}
$$

Or we could have used the 2nd raw moment:

$$
\begin{aligned}
\mathrm{E}[X^2] &= p \cdot 2 \cdot 10^2 + (1-p) \cdot [(10+\delta)^2 + 10^2] \\
136 + 13^2 &= p \cdot 200 + (1-p) \cdot [200 + 20\delta + \delta^2] \\
305 &= 200 + 20(1-p)\delta + (1-p)\delta \cdot \delta \\
105 &= 20 \cdot 3 + 3 \cdot \delta \\
\delta &= 15 \Rightarrow p = \boxed{0.8}
\end{aligned}
$$

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4. Losses are a mixture of two exponentials. With probability p come from an exponential distribution with mean 10, while with probability $1-p$ they come from an exponential with mean $10+\delta$.

   If the mean loss amount is 12, and the variance of losses is 156, what is p?

   A. 0.2 B. 0.3 C. 0.4 D. 0.5 E. 0.6

Note the differences between this and the previous problem. Here, we have two exponentials, whereas in the other one we had an exponential and a shifted exponential.

$$
\begin{aligned}
\mathrm{E}[X] &= p \cdot 10 + (1-p) \cdot (10+\delta) \\
12 &= 10 + (1-p)\delta \\
2 &= (1-p)\delta \\
\mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[X \mid \text{Case}]] \\
156 &= p \cdot 100 + (1-p) \cdot (10+\delta)^2 + \delta^2 \cdot p \cdot (1-p) \\
156 &= 100p + 100(1-p) + 20\delta(1-p) + \delta^2(1-p) + \delta p \cdot \delta(1-p) \\
156 &= 100 + 20 \cdot 2 + \delta \cdot 2 + \delta p \cdot 2 \\
16 &= 2\delta + 2(\delta - 2) = 4\delta - 4 \\
\delta &= 5 \Rightarrow p = \boxed{0.6}
\end{aligned}
$$

Or using raw moments:

$$
\begin{aligned}
\mathrm{E}[X^2] &= p \cdot 2 \cdot 10^2 + (1-p) \cdot 2 \cdot (10+\delta)^2 \\
156 + 12^2 &= 200p + (1-p)(200 + 40\delta + 2\delta^2)
\end{aligned}
$$

$$
\begin{aligned}
300 &= 200 + 40(1-p)\delta + 2(1-p)\delta \cdot \delta \\
100 &= 40 \cdot 2 + 2 \cdot 2 \cdot \delta \\
\delta &= 5 \Rightarrow p = \boxed{0.6}
\end{aligned}
$$

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5. Losses are exponential with mean 50. What is the median of those losses that exceed 100?

A. 35 B. 104 C. 135 D. 150 E. 204

$$
\begin{aligned}
0.5 &= \mathrm{P}[X \le t \mid X > 100] = \frac{\mathrm{P}[100 < X \le t]}{\mathrm{P}[X > 100]} \\
0.5 &= \frac{e^{-100/50} - e^{-t/50}}{e^{-100/50}} \\
e^{-t/50} &= 0.5e^{-100/50} \\
\frac{-t}{50} &= \ln(0.5) - 2 \\
t &= \boxed{134.66}
\end{aligned}
$$

Alternatively, by the memoryless property, t is 100 + median of X, so $t = 100 + -50\ln(0.5) = 134.66$

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6. Losses are exponential with median 50. Two losses occur in a year, with independent loss amounts. What is the probability that at least one of those losses exceeds 100?

A. 0.06 B. 0.25 C. 0.44 D. 0.56 E. 0.60

Let X and Y be our two losses.

$$
\begin{aligned}
0.5 &= 1 - e^{-50/\theta} \\
\theta &= \frac{-50}{\ln(0.5)} = 72.135 \\
\mathrm{P}[X > 100 \cup Y > 100] &= 1 - \mathrm{P}[X \le 100, Y \le 100] \\
&= 1 - \left(1 - e^{-100/\theta}\right)^2 \\
&= 1 - \frac{9}{16} = \frac{7}{16} = \boxed{0.4375}
\end{aligned}
$$

---

7. Losses are exponential with median 50. Two losses occur in a year, with independent loss amounts. What is the probability that both of those losses exceeds 100?

A. 0.06 B. 0.25 C. 0.44 D. 0.56 E. 0.60

Again let X and Y be the two losses. From before, $\theta = 72.135$, and

$$\mathrm{P}[X > 100, Y > 100] = \left(e^{-100/\theta}\right)^2 = \frac{1}{16} = \boxed{0.0625}$$

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8. Losses are exponential with median 50. Two losses occur in a year, with independent loss amounts. What is the probability that the sum of those losses exceeds 100?

A. 0.06 B. 0.25 C. 0.44 D. 0.56 E. 0.60

$X + Y \sim \text{Gamma}(\alpha = 2, \theta = 72.135)$, and

$$\mathrm{P}[X + Y > 100] = e^{-100/\theta} + \frac{100}{\theta}e^{-100/\theta} = \boxed{0.5966}$$

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9. Suppose X has a Gamma distribution with parameters $\alpha > 1$ and θ. Find the mode of X.

A. 0 B. $(\alpha - 1)\theta$ C. $\alpha\theta$ D. $(\alpha - 1)\theta^2$ E. $\alpha\theta^2$

$$\begin{aligned}
f(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha}e^{-x/\theta} = cx^{\alpha-1}e^{-x/\theta} \\
f'(x) &= c(\alpha-1)x^{\alpha-2}e^{-x/\theta} - c \cdot \frac{1}{\theta}x^{\alpha-1}e^{-x/\theta} \\
0 &= (\alpha - 1) - \frac{x}{\theta} \\
x &= \boxed{(\alpha-1)\theta}
\end{aligned}$$

Remark: Our critical value is a maximum since $f(x) \to 0$ as $x \to 0$ or $x \to \infty$

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10. [3.F01.37] For watches produced by a certain manufacturer:

(i) Lifetimes follow a single-parameter Pareto distribution with $\alpha > 1$ and $\theta = 4$.
(ii) The expected lifetime of a watch is 8 years.

Calculate the probability that the lifetime of a watch is at least 6 years.

A. 0.44 B. 0.50 C. 0.56 D. 0.61 E. 0.67

From the tables, $\mathrm{E}[X^k] = \dfrac{\alpha\theta^k}{\alpha - k}$ so $8 = \dfrac{4\alpha}{\alpha - 1}$ and thus $\alpha = 2$.

We want $\mathrm{P}[X > 6] = S(6) = 1 - F(6) = 1 - \left(1 - \left(\frac{4}{6}\right)^2\right) = \left(\frac{2}{3}\right)^2 = \boxed{\frac{4}{9}}$

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11. [3.S06.25] Calculate the skewness of a Pareto distribution with $\alpha = 4$ and $\theta = 1{,}000$.

A. Less than 2
B. At least 2, but less than 4
C. At least 4, but less than 6
D. At least 6, but less than 8
E. At least 8

The raw moments are in the tables, giving us

$$\mu = \frac{\theta}{\alpha - 1} = 333.3$$
$$\mu_2' = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = 333{,}333.3$$
$$\mu_3' = \frac{6\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} = 10^9$$

Therefore $\sigma^3 = \mathrm{Var}(X)^{3/2} = \left(E(X^2) - \mu^2\right)^{3/2} = (222{,}222)^{1.5} = 104{,}756{,}560$ and

$$\begin{aligned}\mu_3 &= \mathrm{E}\left[(X-\mu)^3\right] = \mathrm{E}[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3] \\ &= \mathrm{E}[X^3] - 3\mu\mathrm{E}[X^2] + 3\mu^2\mu - \mu^3 \\ &= 740{,}740{,}741\end{aligned}$$

and so the skewness is $740.740/104.757 = \boxed{7.07}$

Later on, we will see that the answer does't depend on θ, so you can set $\theta = 1$ to make the numbers easier to work with.

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12. Losses are a 2-point mixture. 30% of the time, losses come from a Pareto distribution with $\alpha = 3$ and $\theta = 10$, and 70% of the time losses come from a Pareto distribution with $\alpha = 6$ and $\theta = 10$. What is the median loss amount?

A. 1.50 B. 1.53 C. 1.57 D. 1.60 E. 1.64

$$F(x) = 0.3\left[1 - \left(\frac{10}{x+10}\right)^3\right] + 0.7\left[1 - \left(\frac{10}{x+10}\right)^6\right]$$

$$0.5 = 0.3 - 0.3u + 0.7 - 0.7u^2 \quad \text{where } u = \left(\frac{10}{x+10}\right)^3$$

$$0.7u^2 + 0.3u - 0.5 = 0$$

$$u = \left(\frac{10}{x+10}\right)^3 = 0.6576$$

$$x = \boxed{1.5}$$

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13. [M.S05.9] A loss, X, follows a 2-parameter Pareto distribution with $\alpha = 2$ and unspecified parameter θ. You are given:

$$\mathrm{E}\left[X - 100 \mid X > 100\right] = \frac{5}{3}\mathrm{E}\left[X - 50 \mid X > 50\right]$$

Calculate $\mathrm{E}\left[X - 150 \mid X > 150\right]$.

A. 150 B. 175 C. 200 D. 225 E. 250

For a Pareto distribution, the distribution of $(X - d \mid X > d)$ is a Pareto with the same α and with an updated $\theta' = \theta + d$. So we get

$$\frac{\theta + 100}{2-1} = \frac{5}{3}\frac{\theta + 50}{2-1}$$

$$\theta = 25$$

and then $\mathrm{E}[X - 150 \mid X > 150] = (25 + 150)/(2-1) = \boxed{175}$

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14. [M.S05.34] The distribution of a loss, X , is a two-point mixture:

(i) With probability 0.8, X has a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 100$.

(ii) With probability 0.2, X has a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 3000$.

Calculate $\Pr(X \leq 200)$.

A. 0.76 B. 0.79 C. 0.82 D. 0.85 E. 0.88

$$F(200) = 0.8\left[1 - \left(\frac{100}{100+200}\right)^2\right] + 0.2\left[1 - \left(\frac{3{,}000}{3{,}000+200}\right)^4\right]$$

$$= 0.711 + 0.046 = \boxed{0.757}$$

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15. The average height of adult Americans is 176 cm, with a standard deviation of 6 cm, for males, and 163 cm, with a standard deviation of 5 cm, for females. If heights of each group are normally distributed, what is the probability that a randomly selected American male is taller than a randomly selected American female?

A. 0.85 B. 0.88 C. 0.91 D. 0.93 E. 0.95

Let M denote the height of a randomly selected male, and F a randomly selected female. They are independent, so $M-F$ is normal with mean $\mathrm{E}[M]-\mathrm{E}[F] = 176-163 = 13$, and variance $\mathrm{Var}[M]+(-1)^2\mathrm{Var}[F] = 36+25 = 61$.
$\mathrm{P}[M > F] = \mathrm{P}\left[\dfrac{M-F-13}{\sqrt{61}} > \dfrac{0-13}{\sqrt{61}}\right] = 1-\Phi(\dfrac{-13}{\sqrt{61}} = \Phi(1.66) = \boxed{0.95}$

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16. [3.F05.32] For a certain insurance company, 60% of claims have a normal distribution with mean 5,000 and variance 1,000,000. The remaining 40% have a normal distribution with mean 4,000 and variance 1,000,000.

Calculate the probability that a randomly selected claim exceeds 6,000.

A. Less than 0.10
B. At least 0.10, but less than 0.15
C. At least 0.15, but less than 0.20
D. At least 0.20, but less than 0.25
E. At least 0.25

Let X be a random claim amount.

$$\begin{aligned}\mathrm{P}[X > 6{,}000] &= (0.6)\mathrm{P}[N(5{,}000; 1{,}000^2) > 6{,}000] + (0.4)\mathrm{P}[N(4{,}000; 1{,}000^2) > 6{,}000] \\ &= 0.6(1-\Phi(1)) + 0.4(1-\Phi(2)) = 0.6(0.1587) + 0.4(0.0228) \\ &= \boxed{0.104}\end{aligned}$$

Note that the resulting mixture is not a normal random variable. For one thing, it is bi-modal, with the density having one local max at 4,000 and a second one at 5,000, while a normal random variable has a single mode.

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17. Suppose X and Y are independent normals, each with variance 1,000,000, and with $\mathrm{E}[X] = 5{,}000$ and $\mathrm{E}[Y] = 4{,}000$.

Find $\mathrm{P}[0.6X + 0.4Y > 6{,}000]$

A. 0.026 B. 0.052 C. 0.081 D. 0.104 E. 0.123

Here we have a weighted average, which is a type of sum, so

$$\begin{aligned}
\mathrm{E}[0.6X + 0.4Y] &= 0.6 \cdot 5{,}000 + 0.4 \cdot 4{,}000 = 4{,}600 \\
\mathrm{Var}[0.6X + 0.4Y] &= 0.6^2 \mathrm{Var}[X] + 0.4^2 \mathrm{Var}[Y] \\
\mathrm{SD}[0.6X + 0.4Y] &= 721.11 \\
\mathrm{P}[0.6X + 0.4Y > 6{,}000] &= 1 - \Phi\left(\frac{6{,}000 - 4{,}600}{721.11}\right) = 1 - \Phi(1.94) \\
&= \boxed{0.0262}
\end{aligned}$$

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18. [1.S01.33] For Company A there is a 60% chance that no claim is made during the coming year. If one of more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000.

For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000.

Assume that the total claim amounts of the two companies are independent.

What is the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount?

A. 0.180 B. 0.185 C. 0.217 D. 0.223 E. 0.240

Neither A nor B is normally distributed because there is a positive probability of no claim being made. To use what we know about normal distributions, we have to condition on a payment being made for both A and B.

I will use $A > 0$ and $B > 0$ to denote the case when A and B have claims (that is technically wrong as there is a very small probability of the normal variables being negative, but it simplifies the notation).

$$\begin{aligned}
\mathrm{P}[B > A] &= \mathrm{P}[B > A, A > 0] + \mathrm{P}[B > A, A = 0] \\
&= \mathrm{P}[A > 0] \cdot \mathrm{P}[B > A \mid A > 0] + \mathrm{P}[A = 0] \cdot \mathrm{P}[B > A \mid A = 0] \\
&= 0.4 \cdot \mathrm{P}[B > 0] \cdot \mathrm{P}[B > A \mid A, B > 0] + 0.6 \cdot \mathrm{P}[B > 0] \\
&= (0.4)(0.3)\mathrm{P}[B > A \mid A, B > 0] + (0.6)(0.3)
\end{aligned}$$

And when A and B both have claims we have $B - A$ is a normal random variable with parameters

$$\mathrm{E}[B - A \mid A, B > 0] = \mathrm{E}[B \mid B > 0] - \mathrm{E}[A \mid A > 0] = 9{,}000 - 10{,}000 = -1{,}000$$

$$
\begin{aligned}
\text{Var}[B-A \mid A,B>0] &= \text{Var}[B \mid B>0] + (-1)^2\text{Var}[A \mid A>0] \\
&= 2{,}000^2 + 2{,}000^2 = 8{,}000{,}000 \\
\text{SD}[B-A \mid A,B>0] &= 2{,}828 \\
\text{P}[B-A>0 \mid A,B>0] &= \text{P}\left[\frac{B-A-(-1{,}000)}{2{,}828} > \frac{0-(-1{,}000)}{2{,}828} \mid A,B>0\right] \\
&= 1-\Phi\left(\frac{1{,}000}{2{,}828}\right) \\
&= 1-\Phi(0.35) = 1-0.64 = 0.36 \\
\text{P}[B>A] &= 0.4\cdot 0.3\cdot 0.36 + 0.6\cdot 0.3 = \boxed{0.223}
\end{aligned}
$$

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19. Loss amounts are normally distributed, with a 6.68% chance of exceeding 102 and a 2.28% chance of exceeding 105. What is the probability that a randomly selected loss is greater than 95?

A. 0.11 B. 0.37 C. 0.48 D. 0.66 E. 0.91

$1-6.68\% = 0.9332$, and looking that up on the normal table that is a z-value of 1.5. So 102 is 1.5 standard deviations above the mean. $1-0.0228 = 0.9772$ which has a z-value of 2, so 105 is 2 standard deviations above the mean. That means that

$$
\begin{aligned}
105-102 &= 0.5\text{SD}[X] \\
\text{SD}[X] &= 6 \\
\text{E}[X] &= 105 - 2\cdot 6 = 93 \\
\text{P}[X>95] &= \text{P}\left[\frac{X-\text{E}[X]}{\text{SD}[X]} > \frac{95-93}{6}\right] \\
&= 1-\Phi(0.33) = 1-0.63 = \boxed{0.37}
\end{aligned}
$$

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20. Let Y be a mixture of X_1 and X_2, where X_1 is a normal random variable with mean 0 and standard deviation 1, and X_2 is a normal random variable with mean 0 and standard deviation 5. If $\text{P}[Y = X_1] = 0.9$, what is the kurtosis of Y?

Recall that the kurtosis of a normal random variable is 3.

A. 3.0 B. 8.2 C. 12.4 D. 16.5 E. 49.5

$\text{Kurtosis}(X) = \dfrac{\text{E}\left[(X-\mu)^4\right]}{\sigma^4}$ and so since the kurtosis of a normal is 3, $\text{E}[Z^4] = 3$ if Z is a standard normal.

$$
\begin{aligned}
\text{E}[Y] &= \text{E}[\text{E}[Y \mid \text{Case}]] \\
&= \text{P}[Y=X_1]\cdot\text{E}[X_1] + \text{P}[Y=X_2]\cdot\text{E}[X_2] = 0.9\cdot 0 + 0.1\cdot 0 = 0 \\
\text{Var}[Y] &= \text{E}[\text{Var}[Y \mid \text{Case}]] + \text{Var}[\text{E}[Y \mid \text{Case}]]
\end{aligned}
$$

$$
\begin{aligned}
&= 0.9 \cdot \mathrm{Var}[X_1] + 0.1 \cdot \mathrm{Var}[X_2] + 0.9 \cdot 0.1 \cdot (0-0)^2 \\
\mathrm{Var}[Y] &= 3.4 \\
\text{Or: } \mathrm{E}[Y^2] &= \mathrm{E}[\mathrm{E}[Y^2 \mid \text{Case}]] \\
&= \mathrm{P}[Y = X_1] \cdot \mathrm{E}[X_1^2] + \mathrm{P}[Y = X_2] \cdot \mathrm{E}[X_2^2] \\
&= 0.9 \cdot (0^2 + 1^2) + 0.1 \cdot (0^2 + 5^2) = 3.4 \\
\mathrm{Var}[Y] &= 3.4 - 0^2 = 3.4 \\
\mathrm{E}\left[(Y - \mathrm{E}[Y])^4\right] = \mathrm{E}\left[Y^4\right] &= 0.9 \cdot \mathrm{E}\left[X_1^4\right] + 0.1 \cdot \mathrm{E}\left[X_2^4\right] \\
&= 0.9 \cdot 3 + 0.1 \cdot 3 \cdot 5^4 \\
\mathrm{E}\left[Y^4\right] &= 190.2 \\
\mathrm{Kurtosis}(Y) &= \frac{190.2}{3.4^{4/2}} = \boxed{16.45}
\end{aligned}
$$

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21. Losses are lognormal with median 3 and mean 4. What is the variance of a randomly selected loss?

A. 0.6 B. 2.7 C. 4.5 D. 7.4 E. 12.4

The median of the underlying normal is μ, so the median of the lognormal is e^μ. Finding the other moments in the tables,

$$
\begin{aligned}
\text{Median} &= e^\mu = 3 \\
\mathrm{E}[X] &= e^{\mu + \sigma^2/2} = 4 \\
e^{\sigma^2/2} &= \frac{4}{3} \\
\mathrm{E}[X^2] &= e^{2\mu + 2\sigma^2} = (e^\mu)^2 \cdot (e^{\sigma^2/2})^4 \\
&= 9 \cdot \left(\frac{4}{3}\right)^4 = \frac{256}{9} \\
\mathrm{Var}[X] &= \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \\
&= \frac{256}{9} - 16 = \frac{112}{9} = \boxed{12.4}
\end{aligned}
$$

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22. Losses are lognormal with mean 3 and standard deviation 2. What is the probability that a loss that exceeds 1 will be greater than 4?

A. 0.23 B. 0.26 C. 0.37 D. 0.86 E. 0.89

$$
\begin{aligned}
\mathrm{E}[X] &= 3 = e^{\mu + \sigma^2/2} \\
\mathrm{E}[X^2] &= 3^2 + 2^2 = e^{2\mu + 2\sigma^2} \\
2\ln(3) &= 2\mu + \sigma^2 \\
\ln(13) &= 2\mu + 2\sigma^2
\end{aligned}
$$

$$\begin{aligned}
\sigma^2 &= 0.3677 \\
\mu &= 0.915 \\
\mathrm{P}[X > 4 \mid X > 1] &= \frac{\mathrm{P}[X > 4]}{\mathrm{P}[X > 1]} \\
\mathrm{P}[X > 4] &= \mathrm{P}[\ln(X) > \ln(4)] = 1 - \Phi\left(\frac{\ln(4) - \mu}{\sigma}\right) \\
&= 1 - \Phi(0.78) = 0.2177 \\
\mathrm{P}[X > 1] &= \mathrm{P}[\ln(X) > \ln(1)] = 1 - \Phi\left(\frac{\ln(1) - \mu}{\sigma}\right) \\
&= \Phi(1.51) = 0.9345 \\
\mathrm{P}[X > 4 \mid X > 1] &= \frac{0.2177}{0.9345} = \boxed{0.233}
\end{aligned}$$

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23. [4B.F97.26] You are given the following:

- In 1996, losses follow a lognormal distribution with parameters μ and σ.
- In 1997, losses follow a lognormal distribution with parameters $\mu + \ln(k)$ and σ, where $k > 1$.
- In 1996, $100p\%$ of the losses exceed the mean of the losses in 1997.

Determine σ. Note: z_p is the $100p$th percentile of a normal distribution with mean 0 and variance 1.

A. $2\ln(k)$ B. $-z_p \pm \sqrt{z_p^2 - 2\ln(k)}$ C. $z_p \pm \sqrt{z_p^2 - 2\ln(k)}$ D. $\sqrt{-z_p \pm \sqrt{z_p^2 - 2\ln(k)}}$ E. $\sqrt{z_p \pm \sqrt{z_p^2 - 2\ln(k)}}$

Let X denote a 1996 loss amount.

$$\begin{aligned}
\mathrm{P}\left[X > e^{\mu + \ln(k) + \sigma^2/2}\right] &= p \\
\mathrm{P}\left[X \le e^{\mu + \ln(k) + \sigma^2/2}\right] &= 1 - p \\
\mathrm{P}\left[\frac{\ln(X) - \mu}{\sigma} \le \frac{\ln(k) + \sigma^2/2}{\sigma}\right] &= 1 - p \\
z_{1-p} = -z_p &= \frac{\ln(k) + \sigma^2/2}{\sigma} \\
\sigma^2 + 2z_p\sigma + 2\ln(k) &= 0 \\
\sigma &= \frac{-2z_p \pm \sqrt{4z_p^2 - 4 \cdot 2\ln(k)}}{2} \\
&= \boxed{-z_p \pm \sqrt{z_p^2 - 2\ln(k)}}
\end{aligned}$$

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24. Loss amounts have a survival function given by

$$S(x) = \begin{cases} 1 & x < 0 \\ e^{-2x^2} & x \geq 0 \end{cases}$$

What is the average loss amount?

A. $\sqrt{\dfrac{\pi}{8}}$ B. $\sqrt{\dfrac{\pi}{4}}$ C. $\sqrt{\dfrac{\pi}{2}}$ D. $\sqrt{\pi}$ E. $\sqrt{2\pi}$

A standard normal has density $f(x) = e^{-x^2/2}/\sqrt{2\pi}$, and a $N(\mu, \sigma^2)$ has density $\dfrac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$, so one way to integrate e^{-x^2} is to compare with a normal CDF.

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^\infty e^{-2x^2}\,dx \\
&= \sigma\sqrt{2\pi}\int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)}\,dx \qquad \text{for } 2 = 1/(2\sigma^2) \text{ and } \sigma = 1/2 \\
&= \frac{1}{2}\sqrt{2\pi}\cdot \mathrm{P}[Z > 0] \quad \text{for } Z \sim N(0, 1/4) \\
&= \frac{1}{2}\sqrt{2\pi}\cdot\frac{1}{2} = \boxed{\sqrt{\frac{\pi}{8}}}
\end{aligned}$$

Or: X is a Weibull with $\tau = 2$ and $(1/\theta)^\tau = (1/\theta)^2 = 2$ so $\theta = 1/\sqrt{2}$ and $\mathrm{E}[X] = \dfrac{1}{\sqrt{2}}\Gamma(3/2) = \dfrac{1}{\sqrt{2}}\dfrac{1}{2}\sqrt{\pi} = \sqrt{\dfrac{\pi}{8}}$

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25. Loss amounts have a survival function given by

$$S(x) = \begin{cases} 1 & x < 1 \\ e^{-2x^2} & x \geq 1 \end{cases}$$

What is the average loss amount?

A. 0.03 B. 0.77 C. 1.03 D. 1.77 E. 2.41

Now we have

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^1 1dx + \int_1^\infty e^{-2x^2}\,dx \\
&= 1 + \sqrt{\frac{\pi}{2}}\mathrm{P}[Z > 1] \\
&= 1 + \sqrt{\frac{\pi}{2}}\mathrm{P}\left[\frac{Z-0}{1/2} > \frac{1-0}{1/2}\right]
\end{aligned}$$

$$= 1 + \sqrt{\frac{\pi}{2}}\,(1 - \Phi(2)) = \boxed{1.029}$$

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26. The density function of a random variable is proportional to e^{-x^2} for $x \geq 0.5$ and is 0 otherwise. Find $\mathrm{P}[X \geq 1]$

A. 0.08 B. 0.33 C. 0.55 D. 0.76 E. 0.92

$f(x) = ce^{-x^2}$ for $x > 0.5$, where $c = 1/\int_{0.5}^{\infty} e^{-x^2}dx$ so

$$\begin{aligned}
\mathrm{P}[X \geq 1] &= \frac{\int_1^{\infty} e^{-x^2}dx}{\int_{0.5}^{\infty} e^{-x^2}dx} \\
&= \frac{\mathrm{P}[Z > 1]}{\mathrm{P}[Z > 0.5]} \text{ where } Z \text{ is a normal with mean 0 and variance } 1/2 \\
&= \frac{1 - \Phi\left(\frac{1}{1/\sqrt{2}}\right)}{1 - \Phi\left(\frac{0.5}{1/\sqrt{2}}\right)} \\
&= \frac{1 - 0.9207}{1 - 0.7611} = \boxed{0.33}
\end{aligned}$$

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27. Losses are modeled with a lognormal distribution with parameters μ and σ. If the median loss amount is 1.06 and the mean loss amount is 1.08, what is σ?

A. 0.01 B. 0.02 C. 0.04 D. 0.10 E. 0.19

For a lognormal, the median is e^{μ} and the mean is $e^{\mu+\sigma^2/2}$ so

$$\begin{aligned}
e^{\mu} &= 1.06 \\
\mu &= \ln 1.06 = 0.0583 \\
e^{\mu+\sigma^2/2} &= 1.08 \\
\mu + \sigma^2/2 &= \ln 1.06 + \sigma^2/2 = \ln 1.08 \\
\sigma &= \boxed{0.1933}
\end{aligned}$$

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28. Losses are modeled with a lognormal distribution with mean 0.42 and variance 1.65. Find the probability that losses are at least 1.

A. 0.09 B. 0.18 C. 0.27 D. 0.33 E. 0.36

$$e^{\mu+\sigma^2/2} = 0.42 \qquad e^{2\mu+\sigma^2} = 0.42^2$$

$$
\begin{aligned}
e^{2\mu+2\sigma^2} &= 1.65 + 0.42^2 = 1.8264 \\
e^{\sigma^2} &= \frac{1.8264}{0.42^2} \quad \text{by division} \\
\sigma &= 1.5288 \\
\ln 0.42 &= \mu + 1.5288^2/2, \quad \mu = -2.036 \\
\mathrm{P}[X > 1] &= \mathrm{P}[\ln X > 0] \\
&= \mathrm{P}\left[\frac{\ln X - \mu}{\sigma} > \frac{2.036}{1.5288}\right] = 1 - \Phi(1.33) \\
&= \boxed{0.0918}
\end{aligned}
$$
