An Introduction to the Mathematics of Financial Derivatives: Chapter 10

Source Authors: Neftci, Hirsa

Video By: Zak Fischer, FSA, CERA

Overview

- Multiplication Rules
- **Ito's Lemma (Differential Form)**
- 5 Examples
- Ito's Lemma (Integral Form)
- Extensions to Ito's Lemma

Resources for Ito's Lemma

- Review Videos
- Drill Problem Set

Dropping Terms in the Stochastic Differential

If the ratio:

$$\frac{g(\Delta W_k, h)}{h}$$

vanishes in the mean square sense as $h \to 0$, then we consider $g(\Delta W_k, h)$ as negligible in small intervals.

Dropping Terms in the Stochastic Differential

- Informally, we can think about which terms are in the stochastic differential by the following baseball analogy. Assume that dt is like getting 2 strikes. Since $(dW_t)^2 = dt$, think of dW_t as 1 strike.
- Three strikes and you are out (of the Taylor expansion):

$\cdot$	1	dt	$(dt)^2$
1	$1 = 0$ strikes	$dt = 2$ strikes	$(dt)^2 = 4$ strikes
(dW_t)	$dW_t = 1$ strike	$dW_t \cdot dt = 3$ strikes	$dW_t \cdot (dt)^2 = 5$ strikes
$(dW_t)^2$	$(dW_t)^2 = 2$ strikes	$(dW_t)^2 dt = 4$ strikes	$(dW_t)^2 (dt)^2 = 6$ strikes

$$dS_t = a_t dt + \sigma_t dW_t \Rightarrow (dS_t)^2 = \sigma_t^2 dt$$

Ito's Lemma

- Suppose that:
  1a. $F(S_t, t)$ is a twice-differentiable function of t and of the random process S_t
  2a. $dS_t = a_t dt + \sigma_t dW_t$
  3a. a_t and σ_t are well-behaved drift and diffusion parameters
- Then:
  1b. $dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2} dt$
  2b. $dF_t = [a_t \frac{\partial F}{\partial S_t} + \frac{\partial F}{\partial t} + \frac{1}{2} \sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}] dt + \frac{\partial F}{\partial S_t} \sigma_t dW_t$

Uses of Ito's Lemma

1. Provides a tool for obtaining stochastic differentials for functions of random variables
2. Helps with evaluating certain integrals as well. The 4 steps for this process are (for examples, see Q3-4):
   - Guess a function $F(W_t, t)$
   - Use Ito's Lemma to obtain the SDE for $F(W_t, t)$
   - Apply the integral operator to both sides of the SDE and simplify known integrals.
   - Rearrange to solve for the desired integral

1Q: Let W_t be a standard Wiener process with $F(W_t, t) = W_t^2$. Find the stochastic differential for dF_t

1A:
- $dS_t = dW_t \rightarrow a = 0$ and $\sigma = 1$
- Using Ito's Lemma,
$$dF_t = [\underbrace{a_t \frac{\partial F}{\partial S_t}}_{0} + \underbrace{\frac{\partial F}{\partial t}}_{0} + \underbrace{\frac{1}{2}\sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}}_{1}]dt + \tfrac{\partial F}{\partial S_t}\sigma_t dW_t = dt + 2W_t dW_t$$
- $dF_t = dt + 2W_t dW_t$

Example #2

2Q: Let W_t be a standard Wiener process with $F(W_t, t) = 3 + t + e^{W_t}$. Find the stochastic differential for dF_t

2A:

- $dS_t = dW_t \rightarrow a = 0$ and $\sigma = 1$
- Using Ito's Lemma,

$$dF_t = [\underbrace{a_t \frac{\partial F}{\partial S_t}}_{0} + \underbrace{\frac{\partial F}{\partial t}}_{1} + \tfrac{1}{2}\sigma_t^2 \tfrac{\partial^2 F}{\partial S_t^2}]dt + \tfrac{\partial F}{\partial S_t}\sigma_t dW_t$$

$$= [1 + \tfrac{1}{2}\tfrac{\partial^2 F}{\partial S_t^2}]dt + \tfrac{\partial F}{\partial S_t} dW_t = [1 + \tfrac{1}{2}e^{W_t}]dt + e^{W_t} dW_t$$

- $dF_t = [1 + \frac{1}{2}e^{W_t}]dt + e^{W_t} dW_t$

3Q: Evaluate the integral $\int_0^t s dW_s$

3A: It might not be obvious but we can do this if we apply Ito's Lemma with $F(W_t, t) = tW_t$. Then $a = 0$ and $\sigma = 1$.

- $dF_t = [\underbrace{a_t \frac{\partial F}{\partial S_t}}_{0} + \underbrace{\frac{\partial F}{\partial t}}_{W_t} + \underbrace{\frac{1}{2}\sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}}_{0}]dt + \underbrace{\frac{\partial F}{\partial S_t}}_{t}\sigma_t dW_t$
- $dF_t = W_t dt + t dW_t$
- Thus, $\int_0^t dF_s = \int_0^t d[sW_s] = \int_0^t W_s ds + \int_0^t s dW_s$
- Also, $\int_0^t d[sW_s] = tW_t$
- Therefore, $\int_0^t s dW_s = tW_t - \int_0^t W_s ds$

Example #4

4Q: Evaluate the integral $\int_0^t W_s dW_s$

4A: The choice of F is not immediately obvious, but assume $F(W_t, t) = \frac{1}{2}W_t^2$.

- Using Ito's Lemma,
$$dF_t = [\underbrace{a_t \frac{\partial F}{\partial S_t}}_{0} + \underbrace{\frac{\partial F}{\partial t}}_{0} + \underbrace{.5\sigma_t^2 \frac{\partial^2 F}{\partial S_t^2}}_{.5}]dt + \underbrace{\frac{\partial F}{\partial S_t}}_{W_t} \sigma_t \, dW_t$$
- $dF_t = .5dt + W_t dW_t$
- $\int_0^t dF_s = \int_0^t .5ds + \int_0^t W_s dW_s$
- $.5W_t^2 = .5t + \int_0^t W_s dW_s$
- $\int_0^t W_s dW_s = .5W_t^2 - .5t$

Example #5

5Q: Calculate the expected value $E(F(t))$ where $F(t) = e^{\sigma W_t}$.

5A:
- We can see this because $\sigma W_t \sim N(0, \sigma^2 t)$
- Thus, this is simply the expectation of a lognormal random variable!
  - $E(L) = e^{\mu + .5\sigma^2}$
- Therefore, $E(e^{\sigma W_t}) = e^{.5\sigma^2 t}$

Integral Form of Ito's Lemma

- Stochastic differentials are simply a shorthand for Ito integrals over small intervals
- Using the fact that $\int_0^t dF_u = F(S_t, t) - F(S_0, 0)$ and integrating (2b) from Ito's Lemma gives the Integral Form of Ito's Lemma below
- $\displaystyle\int_0^t \sigma_u F_s dW_u = [F(S_t, t) - F(S_0, 0)] - \int_0^t [a_u F_s + F_u + .5 F_{ss} \sigma_u^2] du$
- The equation in the book, like in Ito's Lemma, is not quite right. It has been fixed in the above equation

Two Extensions to Ito's Lemma

- Multivariate settings
- Jumps

Quiz #1

Show how to calculate $E(W_t^4)$ by applying Ito's Lemma to $F = W_t^4$

Pause Video

No peeking!

Quiz #1 - Solution

- Let $F = W_t^4$
- $dF_t = [\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W_t^2}]dt + \frac{\partial F}{\partial W_t} dW_t = 6W_t^2 dt + 4W_t^3 dW_t$
- $d(W_t^4) = 6W_t^2 dt + 4W_t^3 dW_t$
- In integral form, this is given by $W_t^4 = \int_0^t 6W_s^2 ds + \int_0^t 4W_s^3 dW_s$
- Since the Ito integral is a martingale, $E_0(\int_0^t 4W_s^3 dW_s) = 0$
- Taking expectations of the integral form gives:

$$E(W_t^4) = \int_0^t 6E(W_s^2)ds = \int_0^t 6sds = 3t^2$$

Thus, $\boxed{E(W_t^4) = 3t^2}$