B.1 Poisson Processes

B.1.3 What is a Poisson Process?

- What is a Poisson Process?
- How do we do things with it?
- Exercises

Sources

This lesson comes from Ross Chapter 5 section 3, and the Daniel note section 1.1.

What is a Poisson Process?

A Poisson Process with rate λ is:

- A *Stochastic Process*

  an indexed collection of random variables, e.g. $\{X_t,\ t \geqslant 0\}$

  1. Daily close of S&P $\longrightarrow$NOT a Counting process
  2. Points scored by a field goal kicker $\longrightarrow$IS a Counting process $\longrightarrow$NOT a Poisson Process
  3. Number of insurance claims filed $\longrightarrow$IS a Counting process
- A *Counting* Process

  $\{N(t) = \text{events by time } t,\ t \geqslant 0\}$

  1. $N(t) \nearrow$ and $\geqslant 0$
  2. $N(t)$ is a whole number
  3. $N(t+3) - N(t) =$ events counted between t and $t+3$
- $N(t+s) - N(t) \sim \text{Poisson}(\lambda s) \longrightarrow$ *Stationary Increments*
- $N(b) - N(a)$ and $N(d) - N(c)$ are independent if $b \leqslant c$

  $\longrightarrow$ *Independent Increments*

Turns out: $\mathrm{P}[N(h) > 1] = o(h) \longrightarrow N(t)$ increases by one at a time.

Examples

Trains arrive at a station according to a Poisson Process with rate 1 every 10 minutes. What is the probability that at least one train will arrive in the next 20 minutes?

$\mathrm{P}[N(20) - N(0) \geqslant 1] = \mathrm{P}[N(20) - 0 \geqslant 1]$?

$\lambda = 1/10 \longrightarrow N(20) \sim \text{ Poisson } (\lambda \cdot 20) = \text{ Poisson}((1/10) \cdot 20 = 2)$

From the tables: $p_k = e^{-\lambda}\lambda^k/k!$

$$\begin{aligned}\mathrm{P}[N(20) \geqslant 1] &= p_1 + p_2 + p_3 + \cdots \\ &= 1 - \mathrm{P}[N(20) = 0] \\ &= 1 - e^{-(2)}(2)^0/0! = 1 - e^{-2} \approx \boxed{0.865}\end{aligned}$$

What is the Variance of the number of trains in the next 20 minutes?
$N(20) \sim \text{Poisson}(2) \longrightarrow \mathrm{Var}[N(20)] = 2$

Variance of $\text{Poisson}(\lambda t)$ is λt, Expected value of $\text{Poisson}(\lambda t)$ is λt

As $t \nearrow$, $N(t) \sim N(\lambda t, \lambda t) \longrightarrow$ approximate using continuity correction.

Examples

Trains arrive at a station according to a Poisson Process with rate 1 every 10 minutes.

What is the probability that no trains arrive in the next 10 minutes and exactly 2 trains arrive between 10 and 30 minutes from now?

$\mathrm{P}[N(10) = 0, N(30) - N(10) = 2]$?

$(0, 10) \cap (10, 30) = \varnothing \longrightarrow N(10), N(30) - N(10)$ are independent.

$N(10) \sim \text{Poisson}(\lambda \cdot 10) = \text{Poisson}((1/10) \cdot 10 = 1)$

$N(30) - N(10) \sim \text{Poisson}(\lambda \cdot (30 - 10)) = \text{Poisson}((1/10) \cdot 20 = 2)$

$$\begin{aligned} \mathrm{P}[N(10) = 0, N(30) - N(10) = 2] \\ &= \mathrm{P}[N(10) = 0]\mathrm{P}[N(30) - N(10) = 2] \\ &= \left(e^{-1}\right)\left(e^{-2}(2)^2/2!\right) \\ &= e^{-3}2^2/2 \approx \boxed{0.0996} \end{aligned}$$

Exercise 1

Suppose that trains arrive at a station according to a Poisson Process with rate 1 every 10 minutes. What is the probability that exactly 2 trains arrive in the next 10 minutes or that exactly 1 train arrives between 10 and 40 minutes from now?

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Suppose that trains arrive at a station according to a Poisson Process with rate 1 every 10 minutes. What is the probability that exactly 2 trains arrive in the next 10 minutes or that exactly 1 train arrives between 10 and 40 minutes from now?

$$P[N(10)=2 \text{ or } N(40)-N(10)=1]$$

$N(10)$, $N(40)-N(10)$ are Independent

$$P[A \cup B] = P[A] + P[B] - P[A \cap B]$$

$P[A \cap B] \to P[A]P[B]$

$$N(10) \sim \text{Poisson}(1/10 \cdot 10 = 1)$$

$$N(40)-N(10) \sim \text{Poisson}(1/10 \cdot 30 = 3)$$

$$\boxed{\approx 0.306}$$

$$P[N(10)=2] = e^{-1} 1^2/2! = e^{-1}/2$$

$$P[N(40)-N(10)=1] = e^{-3} 3^1/1! = e^{-3} 3$$

Exercise 2

For a tyrannosaur with 10,000 calories stored:

- The tyrannosaur uses calories uniformly at a rate of 10,000 per day. If his stored calories reach 0, he dies.
- The tyrannosaur eats scientists (5,000 calories each) at a Poisson rate of 1 per day.

What is the probability that the Dinosaur dies at time 1?

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If eats 0 by that time

What is the probability that the Dinosaur dies at time 1? day

$$P[N(1)=0] = e^{-1} \boxed{\cong 0.368}$$

$$N(1) \sim \text{Poisson}(1 \cdot 1)$$