B.3 Markov Chains

B.3.2 Classification of States

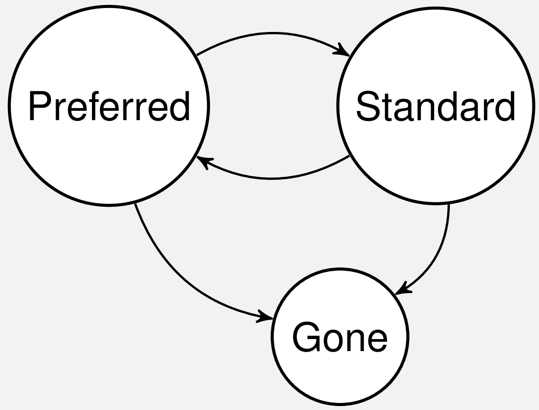
Classifying states

Exercises

Sources

This lesson comes from Ross Chapter 4 section 3.

Classifying states, part 1



No probability of leaving $\longrightarrow$ Gone is an **absorbing** state.

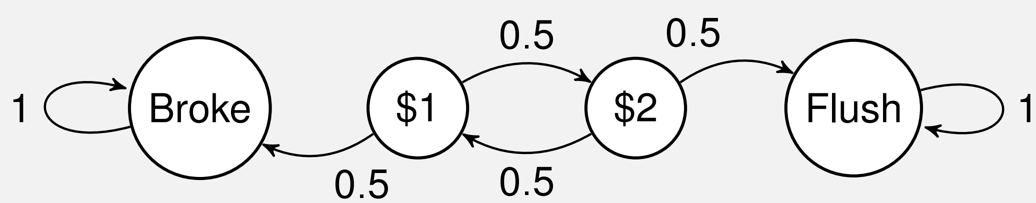
Can be reached in finitely many steps with positive probability
$\longrightarrow$ Gone is **accessible** from Preferred or Standard.

Preferred $\leftrightarrow$ Standard **communicate**. One **class** of states

Gone does not communicate with any state. Another class.

Gambler's Ruin

Suppose a gambler starts with \$1 tosses a coin against an opponent with \$2. If the coin lands heads, the gambler wins \$1, but if the coin lands tails, the gambler losses \$1. The game is repeated until either the gambler is broke or has all \$3.



Nothing is accessible from Broke.

Broke, Flush $\longrightarrow$ absorbing

\$1 and \$2 communicate.

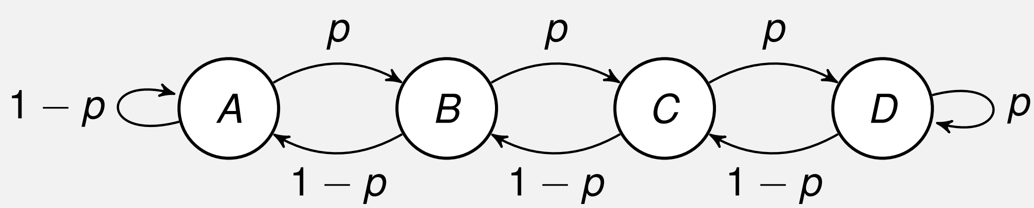
Classes: $\{\text{Broke}\}$ $\{\$1, \$2\}$ $\{\text{Flush}\}$

$$\mathbf{P} = \begin{array}{c} \\ B \\ 1 \\ 2 \\ F \end{array} \!\!\begin{array}{c} \begin{array}{cccc} B & 1 & 2 & F \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

\$1 and \$2 are **Transient** $=$ probability of never returning > 0.

Random Walk

An east/west street has 4 bars. A drunk guy "switches" bars every hour, and always moves east with probability p and west with probability $1 - p$. If he chooses east from the easternmost bar, he just stays put for another hour, and similarly for the westernmost bar.



All states communicate

Classes: $\{A, B, C, D\}$

$\longrightarrow$ **Irreducible** MC

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 1-p & p & 0 & 0 \\ 1-p & 0 & p & 0 \\ 0 & 1-p & 0 & p \\ 0 & 0 & 1-p & p \end{bmatrix} \end{array}$$

$p > 0 \longrightarrow$ all states **Recurrent** $=$ probability of return is 1.

Example

For the following transition matrix of a Markov chain, draw the transition diagram and identify the classes of states. Label each state as recurrent or transient.

$$\mathbf{P} = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Example

For the following transition matrix of a Markov chain, draw the transition diagram and identify the classes of states. Label each state as recurrent or transient.

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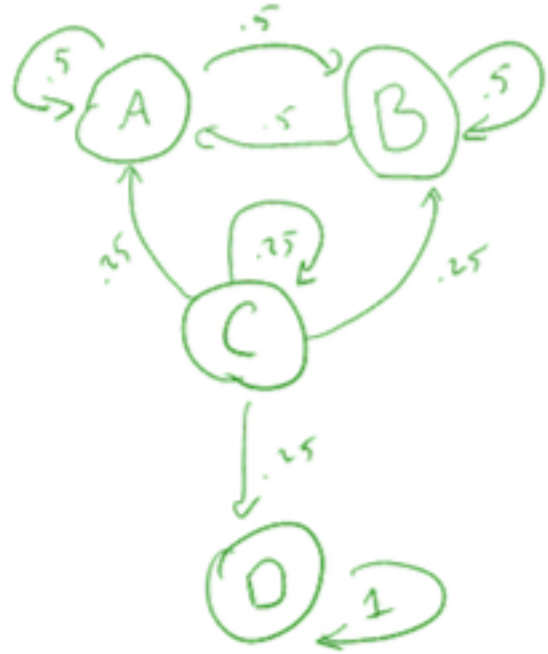
$\{A, B\}$ $\{D\}$ $\{C\}$

recurrent recurrent transient



Exercise - Bonus Malus

An automobile insurance company determines premiums for subsequent years based on the number of accidents in the current year. If an individual has 0 claims this year, the premium goes down by \$100, but it can't fall lower than the minimum premium of \$75. If the number of claims is 1, then the premium goes up \$100, but it's can't be more than the maximum premium of \$275. If the number of claims is 2 or more the premium goes up by \$200, but again to no more than the maximum premium. If the number of claims follows a Poisson distribution with mean 1, construct the Markov chain transition matrix to model this scenario.

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$$\begin{array}{c} \\ 75 \\ 175 \\ 275 \end{array} \begin{array}{c} \begin{array}{ccc} 75 & 175 & 275 \end{array} \\ \begin{bmatrix} e^{-1} & e^{-1} & 1-2e^{-1} \\ e^{-1} & 0 & 1-e^{-1} \\ 0 & e^{-1} & 1-e^{-1} \end{bmatrix} \end{array}$$

$\lambda e^{-\lambda}$