Investment Guarantees Chapter 7: Option Pricing Theory

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Key Exam Topics in This Lesson

- Replication
- The Binomial Model
  - One-Period Binomial Model
  - Two-Period Binomial Model and Dynamic Hedging
- Black-Scholes-Merton
  - BSM Assumptions
  - European Puts and Calls
  - BSM Option Valuation

Replication

Law of One Price

Option prices can be replicated with portfolios containing combinations of risk-free and risky assets

$$\begin{aligned}\text{Option Value} &= \text{Replicating Portfolio Value}\\ &= \text{Risk-Free Asset Amount + Risk Asset Amount}\\ &= ae^{-r} + bS_0\end{aligned}$$

Investment Guarantees – Chapter 7

Replication

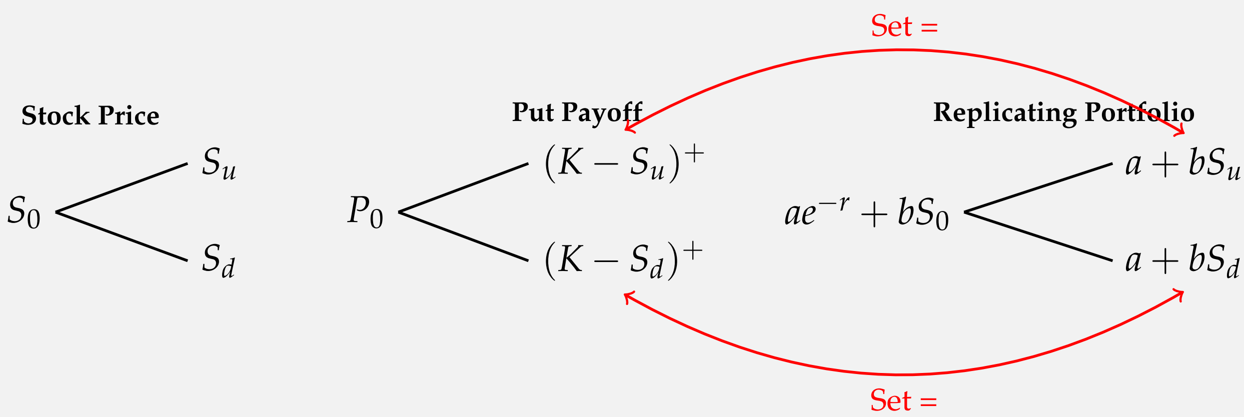
The Binomial Model

One-Period Binomial Model

Two-Period Binomial Model and Dynamic Hedging

Black-Scholes-Merton

One-Period Binomial Model Put Option



You can solve directly for the units of risk-free and risky assets

- This gives you the replicating portfolio value at time 0

One-Period Binomial Q-Measure

We can also value the put option using a risk-neutral probability distribution

$$P_0 = \big(C_u(1 - p^*) + C_d p^*\big)e^{-r}$$

$$p^* = \frac{S_u - S_0 e^r}{S_u - S_d} = \text{Risk-neutral probability that } S \text{ goes } \textbf{down}$$

Requirements for Q-measure

1. Must be equivalent to P-measure: $p_u^* + p_d^* = 1$
2. $E_Q[S_1] = p_u^* S_u + p_d^* S_d = S_0 e^r$

One-Period Binomial Quiz

Your company just issued a single-premium variable annuity based on a stock index that will rise by 10% or fall by 5% over a 3-year period. The VA also guarantees minimum cumulative account growth of 5%. The risk-free rate is 2% all years.

Determine the value of the embedded option in this product as a percentage of premium at issue.

Pause Video

No peeking!

Solution 1 – Calculate Cost of a Put Option Directly

Index Level

$$100 \begin{cases} 110 \\ 95 \end{cases}$$

Put Payoff ($K = 105$)

$$P_0 \begin{cases} (105-110)^+ = 0 \\ (105-95)^+ = 10 \end{cases}$$

First, solve for the risk-neutral probability that the index falls:

$$\begin{aligned} p^* &= \frac{S_u - S_0 e^{3r}}{S_u - S_d} = \frac{110 - 100e^{3(0.02)}}{110 - 95} \\ &= 0.2544 \end{aligned}$$

The put price is:

$$\begin{aligned} P_0 &= \big(C_u(1-p^*) + C_d p^*\big)e^{-3r} \\ &= \big(0 + 10(0.2544)\big)e^{-0.02(3)} \\ &= 2.4 \rightarrow 2.4\% \text{ of premium needed to hedge guarantee} \end{aligned}$$

Solution 2 – Construct a Replicating Portfolio

The company can hedge the guarantee by holding a portfolio of risk-free and risky assets:

$$ae^{-3r} + bS_0$$

To get a and b, set the replicating portfolio equal to the ultimate put payoffs

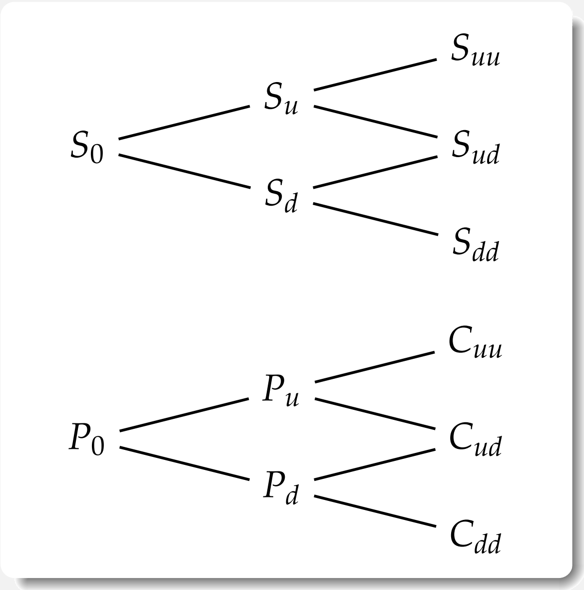
$$\begin{aligned} a + bS_u &= 0 \qquad & a + 110b &= 0 \\ a + bS_d &= K - S_d \qquad & a + 95b &= 10 \end{aligned}$$

This gives us $a = 73.33$ and $b = -2/3$, which means holding a portfolio with

- A long position of $73.33e^{-3(0.02)} = 69.05$ in risk-free assets
- A short position of $-\frac{2}{3}(100) = -66.67$ in the risky index

$$\text{RP Value} = 69.05 - 66.67 = 2.4$$

Two-Period Binomial Model and Dynamic Hedging



Break into 3 one-period models:

1. Solve for the value of P_u and P_d at time 1:

$$P_u = \left(C_{uu}(1 - p_u^*) + C_{ud} p_u^*\right) e^{-r}$$
$$P_d = \left(C_{ud}(1 - p_d^*) + C_{dd} p_d^*\right) e^{-r}$$

2. Solve for the put price at time 0

$$P_0 = \left(P_u(1 - p_0^*) + P_d p_0^*\right) e^{-r}$$

Process is **self-financing**

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Replication

The Binomial Model

Black-Scholes-Merton
- BSM Assumptions
- European Puts and Calls
- BSM Option Valuation

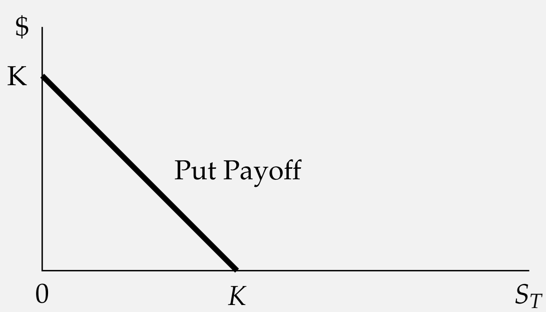
Black-Scholes-Merton Assumptions

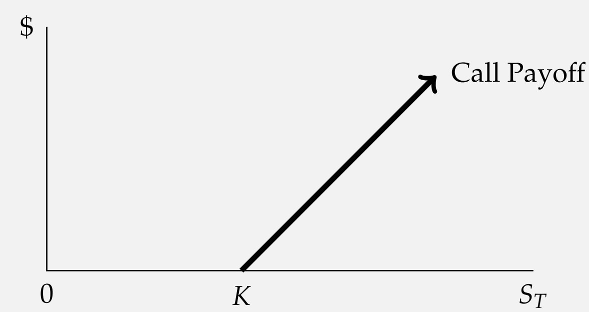
1. S_t follows Geometric Brownian motion (GBM) with constant variance σ^2
   - Lognormal, IID returns
2. Frictionless markets (no transaction costs, taxes)
3. Short selling allowed
4. Continuous trading
5. Interest rates are constant

European Puts and Calls

Put Payoff $= (K - S_T)^+$

Call Payoff $= (S_T - K)^+$





BSM Value of European Puts and Calls

Put-Call Parity

$$Ke^{-r(T-t)} + \mathrm{BSC}_t = S_t + \mathrm{BSP}_t$$

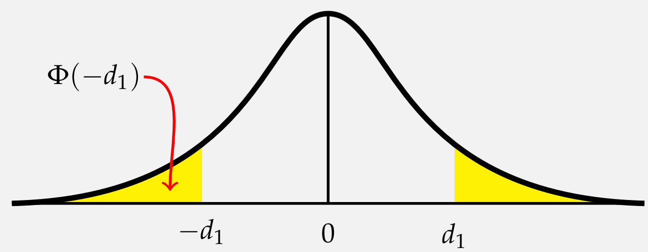
$$\mathrm{BSP}_t = Ke^{-r(T-t)}\Phi(-d_2) - S_t e^{-d(T-t)}\Phi(-d_1)$$

$$\mathrm{BSC}_t = S_t e^{-d(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

$$d_1 = \frac{\ln\frac{S_t}{K} + (T-t)(r-d+\sigma^2/2)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Standard Normal Curve



$$\Phi(-z) = 1 - \Phi(z)$$