The Infinite Actuary Exam P Online Course
TIA SE 1 Solutions
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1. 75% of the customers of ACME Mutual Insurance have auto insurance, and 40% have homeowners insurance. What is the maximum possible probability that a randomly selected customer with auto insurance does not have homeowners insurance?

A. 20% B. 40% C. 60% D. 80% E. 100%

Let A be the set of customers that have auto insurance.
Let H be the set of customers that have homeowners insurance.
If a customer is selected at random and we are told that the selected individual has auto insurance, then the probability that this individual also has homeowners insurance is

$$P[H^C|A] = \frac{P[H^C \text{ and } A]}{P[A]}.$$

We know that the probability of A is 0.75. What, then, is the maximum possible value of $\mathrm{P}[H^c \cap A]$?
Well, $\mathrm{P}[H^c] = 1 - 0.4 = 0.6 < \mathrm{P}[A]$, so, the largest overlap will occur when H^c is wholly contained in A.
In this case, $\mathrm{P}[H^c \cap A] = \mathrm{P}[A] = 0.6$.
Now,

$$P[H^C \mid A] = \frac{P[H^C \text{ and } A]}{P[A]} \leq \frac{0.6}{0.75} = \boxed{0.8},$$

leading to answer choice D.

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2. Suppose Z is a normal random variable with $\mathrm{E}[Z] = 2$ and coefficient of variation 3. Find $\mathrm{P}[Z > 1]$.

A. 0.05 B. 0.34 C. 0.57 D. 0.66 E. 0.95

From the definition of the coefficient of variation, we have

$$3 = \frac{\text{SD of } Z}{EZ} = \frac{\text{SD of } Z}{2}$$

So, SD of $Z = 6$.
Having the mean and standard deviation of Z in hand, we know how to transform it into a standard normal, and then use the normal table to finish.

$$\begin{aligned} P[Z > 1] &= P\left[\frac{Z-2}{6} > \frac{1-2}{6}\right] \\ &= P[\text{Standard Normal} > -1/6] \\ &= 1 - \Phi(-1/6) \\ &= 1 - (1 - \Phi(1/6)) = \Phi(1/6) \approx 0.57 \end{aligned}$$

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3. Let X and Y represent the lifetimes in hours of two components in an electronic device, rounded up to the nearest hour. The joint distribution of X and Y is uniform on possible values, subject to the constraints that X and Y are both positive, and their sum is less than 6. What is the variance of X?

A. 1 B. 5/4 C. 4/3 D. 3/2 E. 2

There are 10 possible values for (X, Y), namely $(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(3,1),(3,2)$ and $(4,1)$. In particular, $\mathrm{P}[X=1]=4/10, \mathrm{P}[X=2]=3/10, \mathrm{P}[X=3]=2/10$ and $\mathrm{P}[X=4]=1/10$, so

$$\begin{aligned}
\mathrm{E}[X] &= 0.4\cdot 1 + 0.3\cdot 2 + 0.2\cdot 3 + 0.1\cdot 4 \\
\mathrm{E}[X] &= 2 \\
\mathrm{E}[X^2] &= 0.4\cdot 1^2 + 0.3\cdot 2^2 + 0.2\cdot 3^2 + 0.1\cdot 4^2 \\
\mathrm{E}[X^2] &= 5\mathrm{Var}[X] \qquad = \mathrm{E}[X^2] - (\mathrm{E}[X])^2 = 5 - 2^2 \\
\mathrm{Var}[X] &= \boxed{1}
\end{aligned}$$

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4. The probability that Rafael Nadal wins a tennis match in straight sets is 70%. Assuming that the outcome of each match is independent, what is the probability that in his next 7 matches that he will win in straight sets at least 5 times?

A. 0.13 B. 0.33 C. 0.44 D. 0.65 E. 0.96

Let X be the number of times that Nadal wins in straight sets. Then X is the sum of 7 independent trials, in which each trial will be 1 with probability 0.7, and 0 with probability 0.3. That means X is a binomial random variable with $n = 7$ and $p = 0.7$.
Now we can compute.

$$\begin{aligned}
P[X \geq 5] &= P[X=5] + P[X=6] + P[X=7] \\
&= \binom{7}{5}(0.7)^5(0.3)^2 + \binom{7}{6}(0.7)^6(0.3)^1 + (0.7)^7 \\
&= 0.318 + 0.247 + 0.082 \approx \boxed{0.65},
\end{aligned}$$

leading to answer choice D.

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5. Suppose that Z be a normal random variable with mean 5 and variance σ^2. Let $f_Z(z,\sigma)$ denote the density of z for a given σ, and let $F_Z(z,\sigma)$ the CDF of Z for a given σ. Which of followings is increasing as σ increases?

A. $f_Z(5,\sigma)$
B. $F_Z(3,\sigma)$
C. $F_Z(8,\sigma)$
D. The 25th percentile of Z
E. None of the above

Let's break down each choice:

A. $f_Z(5,\sigma) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}e^{\frac{-(5-\mu)^2}{2\sigma^2}} = \frac{c}{\sigma}$ which is decreasing as σ increases.

B. $F_Z(3,\sigma) = \phi\left(\frac{3-5}{\sigma}\right) = \phi\left(\frac{-2}{\sigma}\right)$. As σ increases, $\frac{-2}{\sigma}$ will go to 0, and $F_Z(3,\sigma)$ will increase to $\frac{1}{2}$.

C. For $F_Z(8,\sigma) = \phi\left(\frac{8-5}{\sigma}\right)$ will decrease down to $\phi(0) = \frac{1}{2}$.

D. The 25th percentile will be roughly $5 - 0.67\sigma$ and will go to $-\infty$ as σ increases. So it decreases rather than increases.

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6. X and Y are integer valued random variables such that for x and y integers, the joint cdf of X, Y is given by

$$F(x,y) = 1 - e^{-x} - (1 - e^{-x(y+1)})/(y+1)$$

for $x \geq 0, y \geq 0$. Find $\mathrm{P}[1 < X \leq 2, 1 < Y \leq 3]$.

A. 0.05 B. 0.17 C. 0.19 D. 0.41 E. 0.61

Recall that $F(s,t) = P[X \leq s, Y \leq t]$. The fact that X and Y must be integers means we want $\mathrm{P}[X = 2, Y = 2 \text{ or } 3]$. $F(2,3) - F(1,3) = \mathrm{P}[X \leq 2, Y \leq 3] - \mathrm{P}[X \leq 1, Y \leq 3] = \mathrm{P}[X = 2, Y \leq 3]$. That's almost what we want, but includes $\{X = 2, Y \leq 1\}$. We can use a similar expression to take that out:

$\mathrm{P}[X = 2, Y \leq 1] = F(2,1) - F(1,1)$.

Combining, we can say

$$\begin{aligned} \mathrm{P}[X = 2, 1 < Y \leq 3] &= \left[F(2,3) - F(1,3)\right] - \left[F(2,1) - F(1,1)\right] \\ &= F(2,3) - F(1,3) - F(2,1) + F(1,1) \\ &\approx \boxed{0.05} \end{aligned}$$

leading to answer choice A.

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7. A fair die is rolled repeatedly. Let X be the number of rolls needed to obtain 6, and let Y be the number of rolls needed to obtain an even number.

   Find $\mathrm{E}[X \mid Y = 5]$.

   A. 5 B. 6 C. 7 D. 8 E. 9

Method 1: Start with the definition, and switch to the survival function method.

$$\begin{aligned} E[X \mid Y = 5] &= \sum_x xP[X = x \mid Y = 5] \\ &= \sum_{n=1}^{\infty} P[X \geq n \mid Y = 5]. \end{aligned}$$

Notice that it is possible that X is the same as Y, since 6 is an even number, but that X cannot be smaller than Y. Therefore, $P[X \geq n \mid Y = 5] = 1$ for $n = 1, 2, 3, 4$ and 5.
Since we know $Y = 5$, then the 6th roll is an even number. If the 6th roll is a 6, then $X = 5$. Given that we roll an even number, the probability that this number is a 6 is 1 in 3. Thus, $P[X \geq 6 \mid Y = 5] = 2/3$, as it is the complementary event.
Next, $P[X \geq 7 \mid Y = 5] = P[X \neq 5 \text{ and } X \neq 6] = 2/3 \cdot 5/6$.
One more to find the pattern: $P[X \geq 8 \mid Y = 5] = P[X \neq 5 \text{ and } X \neq 6 \text{ and } X \neq 7] = 2/3 \cdot 5/6 \cdot 5/6$.
Now we can compute our sum:

$$\sum_{n=1}^{\infty} P[X \geq n | Y - 5] = 1 + 1 + 1 + 1 + 1 + 2/3 \cdot 5/6 + 2/3 \cdot (5/6)^2 + \cdots$$

With what we know of geometric series (see the one that starts at term 6?), we now have

$$\sum_{n=1}^{\infty} P[X \geq n \mid Y - 5] = 5 + \frac{2/3}{1 - 5/6} = \boxed{9},$$

leading to answer choice E.
Method 2: Split cases according to whether $X = 5$ or not.

$$\begin{aligned} E[X \mid Y = 5] = &\, E[X \mid Y = 5, X = 5]P[X = 5 \mid Y = 5] \\ &+ E[X \mid Y = 5, X \geq 6]P[X > 5 \mid Y = 5] \end{aligned}$$

If we are given that $X = 5$, then $E[X \mid Y = 5, X = 5]$ had better be 5.
As we mentioned above, $P[X = 5 \mid Y = 5] = 1/3$, and $P[X > 5 \mid Y = 5] = P[X \geq 6 \mid Y = 5] = 2/3$.
Finally, given $X > 5$, X is at least 6, so $(X - 5 \mid X > 5)$ is at least 1. The memoryless property says that $(X - 5 \mid X > 5)$ is a geometric starting at 1, with the same p as X, namely $p = 1/6$, so $E[X - 5 \mid X > 5] = 1/(1/6) = 6$, and $E[X \mid X > 5] = E[X - 5 \mid X > 5] + 5 = 5 + 6$.
Putting it all together, $E[X \mid Y = 5] = 1/3 \cdot 5 + 2/3 \cdot (5 + 6) = 5 + 2/3 \cdot 6 = \boxed{9}$

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8. Suppose that X and Y are random variables with $\mathrm{E}[X] = \mathrm{E}[Y] = 1$ and $\mathrm{Var}[X] = \mathrm{Var}[Y] = 4$. If $\mathrm{Cov}(X,Y) = 3$, find $\mathrm{Var}[3X - 2Y > 0]$.

A. 0 B. 8 C. 16 D. 34 E. 52

$$\begin{aligned}\mathrm{Var}(2X - 3Y) &= 9\mathrm{Var}(X) - 2 \cdot 3 \cdot 2 \cdot \mathrm{Cov}(X,Y) + (-2)^2\mathrm{Var}(Y) \\ &= 9 \cdot 4 - 12 \cdot 3 + 4 \cdot 4 = \boxed{16}\end{aligned}$$

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9. Suppose that I roll two independent dice, one red and one blue. Let A be the event that the blue die is even, B the event that the red die is even, and C the event that the sum is even. Which of the following is true?

A. None of them are independent.
B. A and B are pairwise independent, but neither is pairwise independent of C.
C. A and C are pairwise independent, as are B and C, but A and B are not pairwise independent.
D. All three pairs are pairwise independent, but it is not true that all three are mutually independent.
E. All three are mutually independent.

We can easily see that both $P[A]$ and $P[B]$ are 1/2.
To see that $P[C] = 1/2$ as well, note that this is true regardless of what happens with the first die. If the first die is even, then there are three possibilities for the second die to be even, so C will occur half of the time. On the other hand, if the first die is odd, then there are three possibilities for the second die to be odd and form an even sum, so C will still occur half of the time.
Events A and B are independent because

$$P[A \cap B] = P[\text{Red die is even AND Blue die is even}] = 1/2 \cdot 1/2 = P[A]P[B]$$

So answer choice A is false, and answer choice C is also false.
Next, consider that

$$\begin{aligned}P[A \cap C] &= P[\text{Red die is even AND Sum is even}] \\ &= P[\text{Red die is even AND blue die is even}] \\ &= 1/2 \cdot 1/2 = P[A]P[C].\end{aligned}$$

So events A and C are also pairwise independent, eliminating answer choice B. In fact, events B and C are also pairwise independent by a similar argument, but both of our remaining answer choices already contain this information, so we know it must be true.
Finally, consider

$$\begin{aligned}P[A \cap B \cap C] &= P[\text{Red die is even AND Blue die is even AND sum is even}] \\ &= P[\text{Red die is even AND Blue die is even}] \\ &= 1/2 \cdot 1/2 \neq P[A]P[B]P[C].\end{aligned}$$

This means that the events are not mutually independent, which confirms answer choice D and eliminates E simultaneously.

10. Beedle is a salesman, and each month has a chance of getting a bonus of 20. During the 3 summer months, he has a 40% chance of getting a bonus, while during the other months he only has a 30% chance of getting a bonus. Whether or not he gets a bonus in a month is independent of what happened during other months. What is the variance of sum of Beedle's monthly bonuses, summed over a whole year?

A. 52 B. 56 C. 60 D. 1044 E. 1125

The number of bonuses he will receive during the summer is binomial with $n = 3$ and $p = 0.4$. The number of non-summer bonuses is binomial with $n = 9$ and $p = 0.3$. If N is the total number of monthly bonuses he gets, then (using the fact that the binomial variance is $np(1-p)$):

$$\begin{aligned}
\mathrm{Var}[N] &= 3(0.4)(1-0.4) + 9(0.3)(1-0.3) \\
\mathrm{Var}[N] &= 2.61 \\
\mathrm{Var}[20N] &= 20^2 \cdot 2.61 \\
&= \boxed{1044}
\end{aligned}$$

11. A small shipping company has 5 trucks. The number of accidents that each truck has per year has a Poisson distribution with mean 1.5. If each accident costs the company \$250, and the number of accidents per truck are independent, what is the standard deviation of the annual cost of accidents to the shipping company?

A. 685 B. 839 C. 1022 D. 1286 E. 1531

Let N denote the total number of accidents. The distribution of N is $N \sim Poisson(5 \times 1.5)$ because there are 5 trucks. We want to know the standard deviation of the annual cost; the annual cost is $\text{Cost} = 250N$ and the variance is $Var[\text{Cost}] = 250^2 Var[N] = 250^2(7.5) = 468,750$. That means that our standard deviation is $SD[\text{Cost}] = \sqrt{468,750} = \boxed{684.7}$

12. A student who is taking a 30 question multiple choice test knows the answer to 24 of the questions. Whenever the student doesn't know the answer to a question, he chooses uniformly from one of the 5 choices. Given that the student gets a randomly chosen question right, what is the probability that the student guessed on the question?

A. 1/25 B. 1/21 C. 1/18 D. 1/11 E. 1/5

We are interested in determining, for a question chosen at random,

$$P[\text{Guessed}|\text{Answered correctly}].$$

Use the definition of conditional probability to obtain

$$P[\text{Guessed}|\text{Answered correctly}] = \frac{P[\text{Guessed AND Answered correctly}]}{P[\text{Answered correctly}]}.$$

In this case the intersection is easier. We can just condition the "other" way.

$$\begin{aligned}&P[\text{Guessed AND Answered correctly}]\\&= P[\text{Answered correctly}|\text{Guessed}]P[\text{Guessed}]\\&= 1/5 \cdot 6/30.\end{aligned}$$

To find the total probability that he answered the randomly chosen question correctly, note that

$$\begin{aligned}P[\text{Answered correctly}] = &P[\text{Guessed AND Answered correctly}]\\&+ P[\text{He Did NOT Guess AND Answered correctly}].\end{aligned}$$

The first of these we know. The second is easy, since he correctly answers each of the 24 questions on which he does not guess. Putting it all together, then,

$$\begin{aligned}P[\text{GuessedAnswered correctly}] &= \frac{P[\text{Guessed AND Answered correctly}]}{P[\text{Answered correctly}]}\\&= \frac{6/30 \cdot 1/5}{6/30 \cdot 1/5 + 24/30} = \frac{0.04}{0.84} = \boxed{\frac{1}{21}},\end{aligned}$$

leading to answer choice B.

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13. An insurance company sells exactly two types of insurance, home owners and auto insurance. 55% of their customers have homeowners insurance and 30% have both types of insurance. What fraction of the customers have auto insurance?

A. 15% B. 25% C. 70% D. 75% E. 85%

Let A denote the number of customers who have auto insurance and let H denote the number of customers who have homeowners insurance. We are told that 30% of customers have both types of insurance, P[A and H] = .3. We are also told that 55% of customers have homeowners. Since we know 30% of the 55% is customers who have both, the number of customers who have only homeowners insurance is P[H only] = .55 − .3 = .25. Now, we can find the fraction of customers who have auto insurance:

$$\begin{aligned}P[A] + P[A'] &= 1\\P[A] + .25 &= 1\\P[A] &= \boxed{0.75}\end{aligned}$$

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14. If X, Y, and Z are i.i.d. Poisson random variables with mean 3, what is $\mathrm{E}[(X + Y + Z)^2]$?

A. 27 B. 54 C. 63 D. 90 E. 121

The sum of independent Poissons is Poisson, so $X + Y + Z$ is Poisson with mean $3 \cdot 3 = 9$. The variance of a Poisson equals its mean, so is also 9, and the second moment is $9^2 + 9 = \boxed{90}$

Or because X is Poisson, $E[X] = 3 = \text{Var}[X]$. $\text{E}[X^2] = \text{Var}[X] + (\text{E}[X])^2 = 3 + 9 = 12$. Now we can use the independence and compute:

$$\begin{aligned} E[(X+Y+Z)^2] &= E[X^2 + Y^2 + Z^2 + 2XY + 2XZ + 2YZ] \\ &= E[X^2] + E[Y^2] + E[Z^2] + 2(E[XY] + E[XZ + E[YZ]]) \\ &= 3E[X^2] + 2 \cdot 3E[X]E[Y] \\ &= 3 \cdot 12 + 2 \cdot 3 \cdot 3 \cdot 3 = \boxed{90} \end{aligned}$$

Answer choice $\boxed{D}$

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15. Let U and V be independent uniform random variables on the set $\{1, 2, 3, 4, 5\}$. Find $\text{P}[\min\{U, V\} \leq 2 \mid \max\{U, V\} > 2]$

A. 2/5 B. 4/7 C. 2/3 D. 8/11 E. 12/25

There are 12 cases that work: $U \in \{1, 2\}$ and $V \in \{3, 4, 5\}$, which is 6 cases, and another 6 from swapping U and V. There are 4 cases in which $\max\{U, V\} \leq 2$ (namely both U and V are 1 or 2, so 21 cases in which $\max\{U, V\} > 2$. As all cases are equally likely (by the uniform statement), we get 12/21 as our final answer.

Or using the definition of conditional probability,

$$P[\min\{U, V\} \leq 2 \mid \max\{U, V\} > 2] = \frac{P[\min\{U, V\} \leq 2 \text{ and} \max\{U, V\} > 2]}{P[\max\{U, V\} > 2]}$$

For maximums, we prefer less than or equal statements, so

$$\begin{aligned} P[\max\{U, V\} > 2] &= 1 - P[\max\{U, V\} \leq 2] \\ &= 1 - P[U \leq 2, V \leq 2] \\ &= 1 - (2/5) \cdot (2/5) = 21/25 \end{aligned}$$

In order that the minimum be less than 2 while the max is greater than 2, either U is small and V is large or vice versa.

$$\begin{aligned} P[\min\{U, V\} \leq 2 \text{ and} \max\{U, V\} > 2] &= P[U \leq 2, V > 2] + P[U > 2, V \leq 2] \\ &= \frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5} = \frac{12}{25} \end{aligned}$$

Finally,

$$\begin{aligned} P[\min\{U, V\} < 2 \mid \max\{U, V\} > 2] &= \frac{12/25}{21/25} \\ &= \frac{12}{21} = \boxed{frac47} \end{aligned}$$

which is answer choice B.

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16. Suppose that X and Y are independent, Poisson random variables with $\mathrm{E}[X] = 2$ and $\mathrm{E}[Y] = 2.8$. Find $\mathrm{P}[X + Y < 3]$.

A. 0.14 B. 0.15 C. 0.21 D. 0.25 E. 0.29

The key idea for this problem is to remember that independent Poisson random variables sum to a Poisson whose mean is the sum of the means. In this case that means $X + Y \sim \text{Poisson}(2 + 2.8)$.
Now it's easy.

$$\begin{aligned} P[X+Y<3] &= e^{-\lambda}[1 + \lambda + \lambda^2/2] \\ &= 0.0082 \cdot (1 + 4.8 + 4.8^2/2) \\ &= 0.0082 \cdot 17.31 \\ &\approx \boxed{0.14} \end{aligned}$$

which is answer choice A.

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17. Let N be a randomly chosen integer with $1 \leq N \leq 1,000$. What is the probability that N is not divisible by $7, 11$, or 13?

A. 0.66 B. 0.69 C. 0.72 D. 0.75 E. 0.78

We seek the probability of the complement of a union of 3 sets, the set of integers less than 1000 that are divisible by 7 (Set A), 11 (Set B), or 13 (Set C). To compute the probability of this union, we use the inclusion-exclusion principle with three sets:

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C]$$

Since 7, 11, and 13 share no common divisors (other than 1), it follows that in order for a number to be divisible by all three, it would need to be divisible by the product of all three. But $7 \cdot 11 \cdot 13 = 1001$, which means that no number of size 1000 or below can be easily divisible by all three numbers. So $P[A \cap B \cap C] = 0$.
Similarly, any number that is to be divisible by any two of 7,11, and 13 must be divisible by the respective product of the two. From this it follows that

$$\begin{aligned} P[A] &= (\lfloor 1000/7 \rfloor)/1000 = 0.142 \\ P[B] &= (\lfloor 1000/11 \rfloor)/1000 = 0.090 \\ P[C] &= (\lfloor 1000/13 \rfloor)/1000 = 0.076 \\ P[A \cap B] &= (\lfloor 1000/(7 \cdot 11) \rfloor)/1000 = 0.012 \\ P[A \cap C] &= (\lfloor 1000/(7 \cdot 13) \rfloor)/1000 = 0.010 \\ P[B \cap C] &= (\lfloor 1000/(11 \cdot 13) \rfloor)/1000 = 0.006 \end{aligned}$$

Now $P[A \cup B \cup C] = 0.280$, so our answer is $1 - 0.280 = \boxed{0.720}$, answer choice C.

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18. Insurance losses L in a given year have a lognormal distribution with $L = e^X$, where X is a normal random variables with mean 3.9 and standard deviation 0.8. If a \$100 deductible and a \$50 benefit limit are imposed, what is the probability that the insurance company will pay the benefit limit given that a loss exceeds the deductible?

A. 0.10 B. 0.27 C. 0.43 D. 0.66 E. 0.88

In order to pay the benefit limit of \$50, the losses L must be at least \$50 more than the \$100 deductible, or $L \geq 150$. Being given that $L > 100$,

$$\begin{aligned}
P[L \geq 150 | L > 100] &= \frac{P[L \geq 150 \text{ and } L > 100]}{P[L > 100]} \\
&= \frac{P[L \geq 150]}{P[L > 100]} \\
&= \frac{P[e^X \geq 150]}{P[e^X > 100]} \\
&= \frac{P[X \geq \ln(150)]}{P[X > \ln(100)]} \\
&= \frac{P\left[\frac{X-3.9}{0.8} \geq \frac{\ln(150)-3.9}{0.8}\right]}{P\left[\frac{X-3.9}{0.8} > \frac{\ln(100)-3.9}{0.8}\right]} \\
&= \frac{P\left[\frac{X-3.9}{0.8} \geq 1.39\right]}{P\left[\frac{X-3.9}{0.8} > 0.88\right]} \\
&= \frac{1 - \Phi(1.39)}{1 - \Phi(0.88)} \approx \boxed{0.43},
\end{aligned}$$

answer choice C.

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19. A fair 6-sided die is rolled 1,000 times. Using a normal approximation with a continuity correction, what is the probability that the number of 3's that are rolled is greater than 150 and less than 180?

A. 0.78 B. 0.81 C. 0.84 D. 0.88 E. 0.95

The number of 3's rolled out of 1,000 trials is a binomial random variable X with $n = 1000$, and $p = 1/6$. The mean, $E[X]$ is 1000/6, and the variance, $\text{Var}(X) = 100 \cdot 1/6 \cdot 5/6$.
This X may be approximated by a normal random variable N with the same mean $\mu_N = 1000/6$ and standard deviation $\sigma_N = 1000 \cdot 1/6 \cdot 5/6$. But the range of this normal random variable is all real numbers, while the range of values for X is just the integers from 0 to 1000. To adjust for this, we approximate

$$P[150 < X < 180] \approx P[150.5 \leq N \leq 179.5].$$

That is, we don't want to include any values of N that would round down to 150 or up to 180, neither of which is included in our event $\{150 < X < 180\}$.
Now we proceed to use our normal table.

$$P[150.5 \leq N \leq 179.5] = P\left[\frac{150.5 - \mu_N}{\sigma_N} \leq \frac{N - \mu_N}{\sigma_N} \leq \frac{179.5 - \mu_N}{\sigma_N}\right]$$

$$= \Phi(1.09) - \Phi(-1.37) \approx \boxed{0.78},$$

answer choice A.

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20. Four red dice and six blue dice are rolled. Assuming that all ten dice are fair six sided dice, and rolls are independent, what is the probability that exactly three of the red dice are even, and exactly two of the blue dice come up ones?

A. 0.05 B. 0.10 C. 0.16 D. 0.21 E. 0.27

There are $\binom{4}{3} = 4$ ways for exactly three red dice to be even, and $\binom{6}{2} = 15$ ways for exactly 2 of the 6 blue dice to come up ones. The probability of each way for exactly three red dice to be even is $(1/2)^3(1/2)^1$, while the probability of each way for exactly 2 of the blue dice to come up ones is $(1/6)^2(5/6)^4$. Our answer, then, is

$$(4 \cdot (1/2)^3(1/2)^1)(15 \cdot (1/6)^2(5/6)^4) \approx \boxed{0.05},$$

answer choice A.

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21. A life insurance company classifies its customers as being either high risk or low risk. If 20% of the customers are high risk, and high risk customers are three times as likely as low risk customers to file a claim, what percentage of claims that are filed come from high risk customers?

A. 30% B. 37% C. 43% D. 54% E. 60%

Let p be the probability of a claim by a low risk customer. Then the probability of a claim by a high risk customer is $3p$.
The probability of a claim from a randomly chosen customer is, then,

$$\begin{aligned} P[\text{claim}] &= P[\text{High}]P[\text{claim}|\text{High}] + P[\text{Low}]P[\text{claim}|\text{Low}] \\ &= \frac{1}{5} \cdot 3p + \frac{4}{5} \cdot p. \end{aligned}$$

Another way to state "the percentage of claims that are filed that come from high risk customers" is "the probability that given a claim is made, the customer making the claim is a high risk customer." We may therefore compute

$$\begin{aligned} P[\text{High}|\text{claim}] &= \frac{\text{P[High and claim]}}{P[\text{claim}]} \\ &= \frac{\frac{1}{5} \cdot 3p}{\frac{1}{5} \cdot 3p + \frac{4}{5} \cdot p} = 3/7 \approx \boxed{0.43}, \end{aligned}$$

leading to answer choice C.

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22. Suppose that $X_1, \ldots X_{100}$ are random variables with $\mathrm{E}[X_i] = 100$ and $\mathrm{E}\left(X_i^2\right) = 10,100$. If $\mathrm{Cov}(X_i, X_j) = -1$ for $i \neq j$, what is $\mathrm{Var}[S]$, where $S = \sum_{i=1}^{100} X_i$.

A. 0 B. 100 C. $1,000$ D. $5,050$ E. $10,000$

The variance of a sum S of 100 random variables X_i may be written

$$\begin{aligned}\mathrm{Var}(S) &= \sum_{1}^{100} \mathrm{Var}(X_i) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j) \\ &= \sum_{1}^{100} \mathrm{Var}(X_i) + 2\sum_{i>j} \mathrm{Cov}(X_i, X_j)\end{aligned}$$

This is because in the index collection for the first covariance sum both $i = 1, j = 2$ and $i = 2, j = 1$, for example, are needed, while in the second covariance sum, only $i = 2, j = 1$ would appear. This means each pair of numbers below 100 only appears one time in the index of the second covariance sum. Thus the number of terms in that sum is $\binom{100}{2}$.
Fortunately, the variance of each X_i and the covariance of each pair X_i, X_j is the same, so

$$\mathrm{Var}(X_i) = E[X_i^2] - (E[X])^2 = 10,100 - 10,000 = 100,$$

and

$$\begin{aligned}\mathrm{Var}(S) &= 100 \cdot 100 + (-1)\binom{100}{2} \cdot 2 \\ &= 100 \cdot 100 - 100 \cdot 99 = 100(100 - 99) = \boxed{100}.\end{aligned}$$

This is answer choice B.

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23. Let X be a Poisson random variable with second moment 6. Find $\mathrm{P}[X = 3]$.

A. 0.06 B. 0.12 C. 0.18 D. 0.20 E. 0.22

For a Poisson random variable, its distribution is determined by the parameter λ, which is the mean of the variable. We are given the second moment, so we need to derive a value for λ from the second moment.

$$\begin{aligned}X &\sim \mathrm{Poisson}(\lambda) \\ E(X) &= \mathrm{Var}(X) = \lambda \\ E[X^2] &= \lambda + \lambda^2 \\ 6 &= \lambda + \lambda^2 \\ 0 &= \lambda^2 + \lambda - 6 \\ 0 &= (\lambda + 3)(\lambda - 2) \\ \lambda &= 2 \text{ or } -3\end{aligned}$$

We know λ must be greater than or equal to 0, so we get $\lambda = 2$. Now, the probability $P[X = 3]$:

$$P[X = 3] = e^{-\lambda}\frac{\lambda^3}{3!}$$

$$= e^{-2}\frac{8}{6}$$
$$= \boxed{0.18}$$

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24. The cdf of a random variable X satisfies

$$F(x) = 1 - \frac{200^2}{(x+200)^2}$$

for $x > 0$. Find P$[50 < X < 300 \mid X > 100]$.

A. 0.22 B. 0.36 C. 0.51 D. 0.64 E. 0.78

Start from the definition, and work toward an expression with the CDF.

$$\begin{aligned} P[50 < X < 300 \mid X > 100] &= \frac{P[50 < X < 300 \ \& \ X > 100]}{P[X > 100]} \\ &= \frac{P[100 < X < 300]}{P[X > 100]} \\ &= \frac{F(300) - F(100)}{1 - F(100)} \\ &= \frac{1 - (\frac{200}{500})^2 - (1 - (\frac{200}{300})^2)}{(\frac{200}{100+200})^2} \approx \boxed{0.64}, \end{aligned}$$

answer choice D.

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25. Loss amounts have a continuous distribution that ranges from 0 to 3. The density of that distribution, when positive, is proportional to the square of the loss amount. Find the 80th percentile of losses.

A. 0.9 B. 1.3 C. 1.8 D. 2.3 E. 2.8

Being proportional to y^2 means that $f_Y(y) = cy^2$ for some constant c. Since the total probability must be 1,

$$1 = \int_0^3 cy^2 dy = \left.\frac{cy^3}{3}\right|_0^3 = 9c,$$

so $c = 1/9$.
To find the 80th percentile,

$$\begin{aligned} P[Y < t] &= 0.8 \\ \int_0^t y^2/9 \, dy &= 0.8 \\ t^3/27 &= 0.8 \\ t^3 &= 27 \cdot 0.8 \end{aligned}$$

$$t \approx \boxed{2.8},$$

which is answer choice E.

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26. Suppose that (X, Y) are uniformly chosen from the set of integers given by $0 \le X \le 3$ and $X \le Y \le X^2$. Find $\mathrm{P}[Y \le 3]$

A. $\frac{1}{4}$ B. $\frac{1}{3}$ C. $\frac{5}{12}$ D. $\frac{1}{2}$ E. $\frac{2}{3}$

Since we are choosing uniformly, we want to know how many cases there are. They are:

$$\begin{aligned} x &= 0 & y &= 0 \\ x &= 1 & y &= 1 \\ x &= 2 & y &= 2, 3, \text{ or } 4 \\ x &= 3 & y &= 3, 4, \ldots, 9 \end{aligned}$$

That is $1+1+3+7 = 12$ cases. Of those, $Y \le 3$ in $1+1+2+1 = 5$ of them, for an answer of $\boxed{5/12}$ or answer C,

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27. If X is a Poisson random variable with $\mathrm{P}[X = 1] = 2.5\mathrm{P}[X = 0]$, then what is the probability that X will be within 1 standard deviation of $\mathrm{E}[X]$?

A. 0.08 B. 0.29 C. 0.47 D. 0.63 E. 0.81

We know the forms for these probabilities for Poisson random variables.

$$\begin{aligned} P[X = 1] &= 2.5P[X = 0] \\ \lambda e^{-\lambda} &= 2.5e^{-\lambda} \\ \lambda &= 2.5 \end{aligned}$$

So the mean and the variance of X are both 2.5, and the standard deviation is approximately 1.58.
Now we need to find $P[2.5 - 1.58 < X < 2.5 + 1.58] = P[X = 1, 2, 3, 4]$. Again, using the Poisson distribution,

$$\begin{aligned} P[X = 1, 2, 3, 4] &= e^{-\lambda}\left(\lambda + \lambda^2/2 + \lambda^3/3! + \lambda^4/4!\right) \\ &= e^{-2.5}\left(2.5 + 2.5^2/2 + 2.5^3/3! + 2.5^4/4!\right) \approx \boxed{0.809}, \end{aligned}$$

which corresponds to answer choice E.

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28. For $0 \le x \le 2$ and $0 \le y \le 2$, the joint probability mass function of X and Y is

$$P[(X,Y) = (x,y)] = c(6 - x - 2y),$$

and the joint probability mass function is 0 otherwise. Find $P[X + Y \ge 3]$.

A. 1/9 B. 1/8 C. 1/6 D. 1/5 E. 1/3

There are 9 possible cases (3 values of X and Y) and the total probability is 1, which gives

$$\begin{aligned}
1 &= c(6-0-2\cdot 0) + c(6-0-2\cdot 1) + c(6-0-2\cdot 2)\\
&\quad + c(6-1-2\cdot 0) + c(6-1-2\cdot 1) + c(6-1-2\cdot 2)\\
&\quad + c(6-2-2\cdot 0) + c(6-2-2\cdot 1) + c(6-2-2\cdot 2)\\
&= 27c\\
c &= \frac{1}{27}\\
P[X+Y \ge 3] &= c(6-2-2\cdot 1) + c(6-1\cdot 2\cdot 2) + c(6-2-2\cdot 2)\\
&= \frac{1}{27}(2+1+0)\\
&= \boxed{\frac{1}{9}}
\end{aligned}$$

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29. The joint probability mass funciton of X and Y is proportional to $(x^2 + 2xy)$ for integers such that $0 \le x \le 2$ and $x \le y \le x+1$ and is 0 otherwise. Find $E[Y \mid X = 1]$.

A. 10/8 B. 11/8 C. 12/8 D. 13/8 E. 14/8

$$\begin{aligned}
P[X = x, Y = y] &= c(x^2 + 2xy) \text{ when non-zero}\\
P[Y = 1 \mid X = 1] &= \frac{P[X=1, Y=1]}{P[X=1]}\\
&= \frac{P[X=1,Y=1]}{P[X=1,Y=1] + P[X=1,Y=2]} \qquad = \frac{c(1^2 + 2\cdot 1\cdot 1)}{c(1^2+2\cdot 1\cdot 1) + c(1^2 + 2\cdot 1\cdot 2)}\\
&= \frac{3}{3+5} = \frac{3}{8}\\
P[Y = 2 \mid X = 1] &= \frac{c(1^2 + 2\cdot 1\cdot 2)}{c(1^2 + 2\cdot 1\cdot 1) + c(1^2 + 2\cdot 1\cdot 2)}\\
&= \frac{5}{3+5} = \frac{5}{8} E[Y \mid X = 1] \qquad = 1\cdot P[Y=1 \mid X=1] + 2\cdot P[Y=2 \mid X=1]\\
&= 1\cdot\frac{3}{8} + 2\cdot\frac{5}{8}\\
&= \boxed{\frac{13}{8}}
\end{aligned}$$

for answer choice $\boxed{D}$

Note that the proportionality constant cancelled, so we didn't have to find it. To do so, we would have summed over all possible cases, set that probability equal to 1, and gotten $c = 1/96$.

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30. Suppose that $X_1, \ldots, X_5$ are i.i.d., uniform random variables on the set $\{3, 4, 5, 6\}$. Let $\overline{X}$ denote the average of X_1 through X_5, and let $\sigma_{\bar{X}}$ and $\mu_{\bar{X}}$ denote the standard deviation and mean of $\overline{X}$. Find the probability that the minimum and maximum of $X_1, \ldots X_5$ both differ from $\mu_{\bar{X}}$ by at most $\sigma_{\bar{X}}$.

A. 0.00000 B. 0.00098 C. 0.00186 D. 0.00274 E. 0.03125

To begin, let's compute $\mu_{\bar{X}}$ and $\sigma_{\bar{X}}$. First, recall that $E[X_1] = \frac{3+6}{2} = 4.5$ and $\text{Var}(X_1) = [(\#\text{Possible values})^2 - 1]/12 = (4^2 - 1)/12 = 15/12$
The X_i are i.i.d., so $\mu_{\bar{X}} = E\left[\frac{X_1+\cdots+X_5}{5}\right] = \frac{1}{5}(5E[X_1]) = E[X_1] = 4.5$, and
$\sigma^2_{\bar{X}} = \text{Var}\left(\frac{X_1+\cdots+X_5}{5}\right) = \frac{1}{5^2}(5 \cdot \text{Var}(X_1)) = \frac{1}{5} \cdot \frac{15}{12} = 0.25$.
Now we can proceed with the question at hand. $\mu_{\overline{X}} - \sigma_{\overline{X}} = 4.5 - \sqrt{0.25} = 4$ and $\mu_{\overline{X}} + \sigma_{\overline{X}} = 4.5 + \sqrt{0.25} = 5$, so we want the min and max to both be 4 or 5. That happens if and only if all 5 of our values are 4 or 5, which has probability $(2/4)^5 = 1/32 = \boxed{0.03125}$ which corresponds to answer choice E.
