The Infinite Actuary Exam STAM Online Course
A.1.7. Sums
Last updated April 11, 2018

1. [4B.S93.9] If X and Y are independent random variables, which of the following statements are true?

   (i) $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

   (ii) $\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

   (iii) $\mathrm{Var}(aX+bY) = a^2\mathrm{E}[X^2] - a(\mathrm{E}[X])^2 + b^2\mathrm{E}[Y^2] - b(\mathrm{E}[Y])^2$

   A. (i) only B. (i) and (ii) only C. (i) and (iii) only D. (ii) and (iii) only E. (i), (ii) and (iii)

Since X and Y are independent, $\mathrm{Cov}(X,Y) = 0$ and $\mathrm{Var}[aX+bY] = a^2\mathrm{Var}[X] + b^2\mathrm{Var}[Y]$. Applying that with $a = b = 1$ gives (i), and with $a = 1, b = -1$ gives (ii), so those are true.
(iii) is false since $a^2\mathrm{Var}[X] = a^2\mathrm{E}[X^2] - a^2(\mathrm{E}[X])^2$, i.e., they are missing the square on the second term.
So the answer is $\boxed{B}$

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2. An insurance company has two types of customers: high risk customers, whose annual loss amounts have mean 20 and variance 50, and low risk customers, whose annual losses have mean 10 and variance 20.

   The loss amounts of the different customers are all independent. In a group with 6 high risk and 4 low risk customers, what is the variance of the total annual losses from the group?

   A. 38 B. 62 C. 289 D. 380 E. 620

Here, we have the sum of 10 independent random variables, 6 with variance 50, and 4 with variance 20. For independent variables, the variance of the sum is the sum of the variances, and we have

$$6 \cdot 50 + 4 \cdot 20 = \boxed{380}$$

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3. I have two fair, but unusual coins. One is gold, and the two sides are marked "1" and "3," while the other is silver with its sides marked "2" and "4." Suppose that I flip both coins. Let W be the average of the two sides that land face up. Let Z be the value that is face up if I choose one of the two coins at random to look at.

   Let $a = \mathrm{P}[Z = 1] - \mathrm{P}[W = 1]$, and let $b = \mathrm{Var}[Z] - \mathrm{Var}[W]$. Find a and b.

   A. $a = 0, b = 0$
   B. $a = 0, b = 0.25$
   C. $a = 0.25, b = 0.25$
   D. $a = 0.25, b = 0.5$
   E. $a = 0.25, b = 0.75$

   Z is 1 if we chose the gold coin, and it lands with the 1 showing, so $\mathrm{P}[Z = 1] = 1/4$. W cannot be 1 as the smallest it can be is 1.5 (when the gold coin is 1 and the silver is 2), so $\mathrm{P}[W = 1] = 0$ and $a = 1/4$.

   In fact, Z is uniform on $\{1, 2, 3, 4\}$, so $\mathrm{Var}[Z] = 1.25$. Each coin individually has variance $2^2 \cdot 0.5^2 = 1$, so since $W = (\text{gold} + \text{silver})/2$, $\mathrm{Var}[W] = \frac{1}{4} \cdot (\mathrm{Var}[\text{gold}] + \mathrm{Var}[\text{silver}]) = 1/4(1 + 1) = 1/2$ and $b = 1.25 - 0.5 = 0.75$ giving us answer choice $\boxed{E}$

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4. [4.F00.32] You are given the following for a sample of five observations from a bivariate distribution:

x	y
1	4
2	2
4	3
5	6
6	4

   (i)
   (ii) $\overline{x} = 3.6, \overline{y} = 3.8$

   A is the covariance of the empirical distribution F_e as defined by these five observations. B is the maximum possible covariance of an empirical distribution with identical marginal distributions to F_e. Determine $B - A$.

   A. 0.9 B. 1.0 C. 1.1 D. 1.2 E. 1.3

The covariance will be maximized if we pair the smallest values together, the 2nd smallest values together, etc., giving

$$A = \mathrm{E}[XY] - \mathrm{E}[X] \cdot \mathrm{E}[Y] = \frac{1 \cdot 4 + 2 \cdot 2 + 4 \cdot 3 + 5 \cdot 6 + 6 \cdot 4}{5} - 3.6 \cdot 3.8 = 1.12$$

$$B = \frac{1 \cdot 2 + 2 \cdot 3 + 4 \cdot 4 + 5 \cdot 4 + 6 \cdot 6}{5} - 3.6 \cdot 3.8 = 2.32$$

$$B - A = \boxed{1.2}$$

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5. [4B.F99.29] You are given the following:

- A is a random variable with mean 5 and coefficient of variation 1.
- B is a random variable with mean 5 and coefficient of variation 1.
- C is a random variable with mean 20 and coefficient of variation 1/2.
- A, B and C are independent.
- $X = A + B$
- $Y = A + C$

Determine the correlation coefficient between X and Y.

A. $\dfrac{-2}{\sqrt{10}}$ B. $\dfrac{-1}{\sqrt{10}}$ C. 0 D. $\dfrac{1}{\sqrt{10}}$ E. $\dfrac{2}{\sqrt{10}}$

$\mathrm{Corr}(X, Y) = \mathrm{Cov}(X, Y)/(\mathrm{SD}[X]\mathrm{SD}[Y])$, so first we need all of those pieces.

$$\begin{aligned}
1 &= \frac{\mathrm{SD}[A]}{\mathrm{E}[A]} = \frac{\mathrm{SD}[A]}{5} \\
\mathrm{Var}[A] &= 25 = \mathrm{Var}[B] \\
\frac{1}{2} &= \frac{\mathrm{SD}[C]}{\mathrm{E}[C]} = \frac{\mathrm{SD}[C]}{20} \\
\mathrm{Var}[C] &= 100 \\
\mathrm{Var}[X] &= \mathrm{Var}[A] + \mathrm{Var}[B] = 25 + 25 = 50 \\
\mathrm{Var}[Y] &= \mathrm{Var}[A] + \mathrm{Var}[C] = 25 + 100 = 125 \\
\mathrm{Cov}[X, Y] &= \mathrm{Cov}[A + B, A + C] \\
&= \mathrm{Cov}[A, A] + \mathrm{Cov}[A, C] + \mathrm{Cov}[B, A] + \mathrm{Cov}[B, C] \\
&= \mathrm{Var}[A] + 0 + 0 + 0 = 25 \\
\mathrm{Corr}[X, Y] &= \frac{25}{\sqrt{50}\sqrt{125}} = \boxed{\frac{1}{\sqrt{10}}}
\end{aligned}$$

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6. Let X and Y be independent Poissons with means 2 and 3, respectively. If $W = 0.4X + 0.6Y$, find $\mathrm{Var}[W] - \mathrm{E}[W]$.

A. -1.2 B. -0.24 C. 0 D. 0.24 E. 1.2

Here we have a weighted average (type of sum), not a mixture.

$$\begin{aligned}
\mathrm{E}[W] &= 0.4\mathrm{E}[X] + 0.6\mathrm{E}[Y] = 0.4 \cdot 2 + 0.6 \cdot 3 = 2.6 \\
\mathrm{Var}[W] &= 0.4^2\mathrm{Var}[X] + 0.6^2\mathrm{Var}[Y] + 2 \cdot 0.4 \cdot 0.6\mathrm{Cov}[X, Y] \\
&= 0.16 \cdot 2 + 0.36 \cdot 3 + 0 = 1.4 \\
\mathrm{Var}[W] - \mathrm{E}[W] &= 1.4 - 2.6 = \boxed{-1.2}
\end{aligned}$$
