The Infinite Actuary Exam STAM Online Course
A.2.2. Gamma Distributions
Last updated April 11, 2018

1. Losses are exponential with median 50. Two losses occur in a year, with independent loss amounts. What is the probability that the sum of those losses exceeds 100?

A. 0.06 B. 0.25 C. 0.44 D. 0.56 E. 0.60

Let X and Y denote the loss amounts. As 50 is the median,

$$
\begin{aligned}
\mathrm{P}[X \le 50] &= 0.5 \\
\mathrm{P}[X > 50] &= 1 - \mathrm{P}[X \le 50] = 1 - 0.5 = 0.5 \\
e^{-50/\theta} &= 0.5 \\
\theta &= \frac{-50}{\ln(0.5)} = 72.135 \\
(X+Y) &\sim \text{Gamma}(\alpha = 2, \theta = 72.135) \\
\mathrm{P}[X+Y > 100] &= e^{-100/\theta} + \frac{100}{\theta} e^{-100/\theta} = \boxed{0.5966}
\end{aligned}
$$

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2. Suppose X has a Gamma distribution with parameters $\alpha > 1$ and θ. Find the mode of X.

A. 0 B. $(\alpha-1)\theta$ C. $\alpha\theta$ D. $(\alpha-1)\theta^2$ E. $\alpha\theta^2$

$$
\begin{aligned}
f(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^\alpha} e^{-x/\theta} = c x^{\alpha-1} e^{-x/\theta} \\
f'(x) &= c(\alpha-1)x^{\alpha-2}e^{-x/\theta} - c \cdot \frac{1}{\theta} x^{\alpha-1} e^{-x/\theta} \\
0 &= (\alpha - 1) - \frac{x}{\theta} \\
x &= \boxed{(\alpha-1)\theta}
\end{aligned}
$$

Remark: Our critical value is a maximum since $f(x) \to 0$ as $x \to 0$ or $x \to \infty$

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3. [4B.F98.27] Determine the skewness of a gamma distribution with coefficient of variation of 1.

A. 0 B. 1 C. 2 D. 4 E. 6

Since skewness is scale invariant, we may assume $\theta = 1$ to simplify the calculations.

$$
\begin{aligned}
\mathrm{CV}[X] &= \frac{\sqrt{\alpha\theta^2}}{\alpha\theta} = \frac{1}{\sqrt{\alpha}} \\
\alpha &= 1 \\
\mathrm{SD}[X] &= \sqrt{\theta^2} = \theta = 1
\end{aligned}
$$

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$$\begin{aligned}
\mathrm{E}[(X-\theta)^3] &= \mathrm{E}[X^3] - 3\theta \mathrm{E}[X^2] + 3\theta^2 \mathrm{E}[X] - \theta^3 \\
&= \mathrm{E}[X^3] - 3\mathrm{E}[X^2] + 2 \\
&= 3! - 3 \cdot 2! + 2 = 2 \\
\mathrm{Skew}[X] &= \frac{2}{1^3} = \boxed{2}
\end{aligned}$$

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4. The probability generating function for a loss amount X satisfies

$$P(z) = \frac{1}{(1-3\ln z)^3}, \quad z < e^{1/3}$$

Find the variance of X.

A. 1 B. 3 C. 9 D. 27 E. 36

We could recognize this PGF:

$$\begin{aligned}
P(z) &= \mathrm{E}[z^X] = \mathrm{E}[e^{X \ln z}] = M(\ln(z)) = M(t) \text{ for } t = \ln(z) \\
P(z) &= \frac{1}{(1-3\ln(z))^3} = \frac{1}{(1-3t)^3}
\end{aligned}$$

which is the MGF of a Gamma with $\alpha = \theta = 3$. Or:

$$\begin{aligned}
P'(z) &= \frac{-3}{(1-3\ln(z))^4} \cdot \frac{-3}{z} = \frac{9}{z(1-3\ln(z))^4} \\
P'(1) &= \mathrm{E}[X] = 9 \\
P''(z) &= \frac{9(-4)(-3)}{(1-3\ln(z))^5}\frac{1}{z^2} + \frac{9}{(1-3\ln(z))^4}\frac{-1}{z^2} \\
P''(1) &= \mathrm{E}[X(X-1)] = \frac{27 \cdot 4}{1} + \frac{-9}{1} = 99 \\
\mathrm{E}[X^2] - \mathrm{E}[X] &= \mathrm{E}[X^2] - 9 = 99 \\
\mathrm{E}[X^2] &= 108, \quad \mathrm{Var}[X] = 108 - 9^2 = \boxed{27}
\end{aligned}$$
