A.2 Key Distributions - Outline

A.2.2 Gamma Distribution
- Density and CDF
- Mean and Variance
- Incomplete Gamma Function
- Exercises

Gamma Density

A Gamma(α, θ) random variable has density

$$\begin{aligned} f(x) &= \frac{(x/\theta)^\alpha e^{-x/\theta}}{x\Gamma(\alpha)} \quad \text{(table version)} \\ &= \frac{x^{\alpha-1}}{\theta^\alpha \Gamma(\alpha)} e^{-x/\theta} \end{aligned}$$

If α is an integer then

$$\begin{aligned} \Gamma(\alpha) &= (\alpha-1)! \\ &= (\alpha-1)\cdot(\alpha-2)! \\ \Gamma(\alpha) &= (\alpha-1)\cdot\Gamma(\alpha-1) \text{ for any } \alpha > 1 \end{aligned}$$

Old exams have given you $\Gamma(\alpha)$ when α is not an integer

$$\begin{aligned} \text{e.g., } \Gamma(1/2) &= \sqrt{\pi} \\ \Gamma(3/2) &= \left(\frac{3}{2}-1\right)\Gamma(1/2) = \frac{1}{2}\sqrt{\pi} \end{aligned}$$

Gamma CDF

An exponential is a Gamma with $\alpha = 1$.
Suppose $\alpha > 0$ is an integer.
(This case is also called the Erlang distribution.)
If $X_1, \ldots, X_\alpha$ are iid exponential(θ) random variables, then
$X_1 + \cdots + X_\alpha \sim \text{Gamma}(\alpha, \theta)$.

- α is the *shape* parameter
- θ is the *scale* parameter.

The exam tables don't really include the CDF. They say that

$$F(x) = \Gamma\left(\alpha; \frac{x}{\theta}\right)$$

If you look up the definition, that says that

$$F(x) = \int_0^x f(t)\,dt$$

which we already knew.

Gamma CDF

If α is an integer, the CDF is nice. Memorize it at least up to $\alpha = 2$.

$$\alpha = 1 : F(x) = 1 - e^{-x/\theta}$$

$$\alpha = 2 : F(x) = 1 - e^{-x/\theta} - \frac{x}{\theta}e^{-x/\theta}$$

$$\alpha = 3 : F(x) = 1 - e^{-x/\theta} - \frac{x}{\theta}e^{-x/\theta} - \frac{(x/\theta)^2}{2}e^{-x/\theta}$$

$$\text{General } \alpha : F(x) = 1 - \sum_{i=0}^{\alpha-1} \mathrm{P}[\text{Poisson}(x/\theta) = i]$$

The survival function is:

$$\alpha = 1 : S(x) = e^{-x/\theta}$$

$$\alpha = 2 : S(x) = e^{-x/\theta} + \frac{x}{\theta}e^{-x/\theta}$$

$$\alpha = 3 : S(x) = e^{-x/\theta} + \frac{x}{\theta}e^{-x/\theta} + \frac{(x/\theta)^2}{2}e^{-x/\theta}$$

Mean and Variance

When α is an integer, a Gamma(α, θ) random variable is the sum of α independent exponential(θ) variables.

So $\mathrm{E}[X] = \alpha \cdot \theta$ and $\mathrm{Var}[X] = \alpha \cdot \theta^2$.

When α is not an integer, those two formulas still hold.

Example

The number of annual losses N satisfies $\mathrm{P}[N = 0] = 0.2$, $\mathrm{P}[N = 1] = 0.5$ and $\mathrm{P}[N = 2] = 0.3$. Loss amounts are independent exponentials with mean 10. What are the mean and variance of the annual loss amounts?
Let S denote the annual loss amount.

$$
\begin{aligned}
\mathrm{E}[S] &= \mathrm{E}[S \mid N = 0] \cdot \mathrm{P}[N = 0] + \mathrm{E}[S \mid N = 1] \cdot \mathrm{P}[N = 1] \\
&\qquad + \mathrm{E}[S \mid N = 2] \cdot \mathrm{P}[N = 2] \\
&= 0 \cdot 0.2 + 10 \cdot 0.5 + 20 \cdot 0.3 = 11 \\
\mathrm{E}[S] &= \mathrm{E}[\mathrm{E}[S \mid N]] = \mathrm{E}[10N] = 10(0 \cdot 0.2 + 1 \cdot 0.5 + 2 \cdot 0.3) = 11 \\
\mathrm{Var}[S] &= \mathrm{E}[\mathrm{Var}[S \mid N]] + \mathrm{Var}[\mathrm{E}[S \mid N]] \\
&= \mathrm{E}[100N] + (0^2 \cdot 0.2 + 10^2 \cdot 0.5 + 20^2 \cdot 0.3 - 11^2) \\
&= 110 + 49 = 159
\end{aligned}
$$

Gamma CDF Revisited

The exam tables list the Gamma CDF as

$$F(x) = \Gamma\left(\alpha; \frac{x}{\theta}\right)$$

If α is an integer, we saw

$$\Gamma\left(\alpha; \frac{x}{\theta}\right) = 1 - \sum_{i=0}^{\alpha-1} e^{-x/\theta} \cdot \frac{(x/\theta)^i}{i!}$$

$\Gamma(\alpha; y) =$ *incomplete Gamma function.*
To evaluate it (only need to do for $\alpha = 1$ or 2, possibly 3),
$\Gamma(\alpha; y) =$ cdf of a Gamma with $\theta = 1$ evaluated at y.

Some other exam table formulas also require the incomplete Gamma function.

Examples

Suppose that X is an inverse Gamma random variable with parameters $\alpha = 3$ and $\theta = 50$. What is $\mathrm{P}[X < 100]$?

From the table, $F(x) = 1 - \Gamma\left(\alpha; \frac{\theta}{x}\right)$.

Note the reversals from the Gamma cdf; in particular, we now have θ/x instead of x/θ.

$$\begin{aligned}
F(100) &= 1 - \Gamma\left(3; \frac{50}{100}\right) = 1 - \Gamma\left(3; \frac{1}{2}\right) \\
&= 1 - \left[1 - \sum_{i=0}^{2} e^{-1/2} \cdot \frac{(1/2)^i}{i!}\right] \\
&= e^{-1/2} + \frac{1}{2}e^{-1/2} + \frac{(1/2)^2}{2!}e^{-1/2} \\
&= \frac{13}{8}e^{-1/2}
\end{aligned}$$

Exercise 1

X is a Gamma random variable with mean 15 and variance 75.
Find $\mathrm{P}[X \leq 10]$.

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$$
\begin{aligned}
\mathrm{E}[X] &= \alpha\theta = 15 \\
\mathrm{Var}[X] &= \alpha\theta^2 = 75 \\
\theta &= \frac{75}{15} = 5 \\
\alpha &= 3 \\
\mathrm{P}[X \leq 10] &= 1 - e^{-10/\theta} - \left(\frac{10}{\theta}\right) e^{-10/\theta} - \dots \\
&= 1 - e^{-10/5} - \left(\frac{10}{5}\right) e^{-2} - \frac{(10/5)^2}{2!} e^{-2} \\
&= 1 - e^{-2}(1 + 2 + 2) \\
&= \boxed{0.3233}
\end{aligned}
$$

Exercise 2

The number of annual losses N satisfies $\mathrm{P}[N = 0] = 0.2$, $\mathrm{P}[N = 1] = 0.5$ and $\mathrm{P}[N = 2] = 0.3$. Loss amounts are independent exponentials with mean 10. What is the probability that annual losses will exceed 15?

Exercise 2

The number of annual losses N satisfies $\mathrm{P}[N = 0] = 0.2$, $\mathrm{P}[N = 1] = 0.5$ and $\mathrm{P}[N = 2] = 0.3$. Loss amounts are independent exponentials with mean 10. What is the probability that annual losses will exceed 15?

Let S denote the annual loss amount.

$$\begin{aligned}
\mathrm{P}[S > 15] &= \mathrm{P}[N = 0] \cdot \mathrm{P}[S > 15 \mid N = 0] \\
&\quad + \mathrm{P}[N = 1] \cdot \mathrm{P}[S > 15 \mid N = 1] \\
&\quad + \mathrm{P}[N = 2] \cdot \mathrm{P}[S > 15 \mid N = 2] \\
&= 0.2 \cdot 0 + 0.5 \cdot e^{-15/10} + 0.3 \cdot \left(1 + \frac{15}{10}\right) e^{-15/10} \\
&= \boxed{0.279}
\end{aligned}$$