**The Infinite Actuary Exam STAM Online Course**
**A.2.4. Normal Distributions**
Last updated April 11, 2018

1. The average height of adult Americans is 176 cm, with a standard deviation of 6 cm, for males, and 163 cm, with a standard deviation of 5 cm, for females. If heights of each group are normally distributed, what is the probability that a randomly selected American male is taller than a randomly selected American female?

   A. 0.85 B. 0.88 C. 0.91 D. 0.93 E. 0.95

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Let $M$ denote the height of a randomly selected male, and F a randomly selected female. They are independent, so $M-F$ is normal with mean $\mathrm{E}[M]-\mathrm{E}[F]=176-163=13$, and variance $\mathrm{Var}[M]+(-1)^2\mathrm{Var}[F]=36+25=61$.

$\mathrm{P}[M>F]=\mathrm{P}\left[\frac{M-F-13}{\sqrt{61}}>\frac{0-13}{\sqrt{61}}\right]=1-\Phi(\frac{-13}{\sqrt{61}}=\Phi(1.66)=\boxed{0.95}$

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2. [3.F05.32] For a certain insurance company, 60% of claims have a normal distribution with mean 5,000 and variance 1,000,000. The remaining 40% have a normal distribution with mean 4,000 and variance 1,000,000.

   Calculate the probability that a randomly selected claim exceeds 6,000.

   A. Less than 0.10
   B. At least 0.10, but less than 0.15
   C. At least 0.15, but less than 0.20
   D. At least 0.20, but less than 0.25
   E. At least 0.25

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Let $X$ be a random claim amount.

$$\begin{aligned}\mathrm{P}[X>6{,}000]&=(0.6)\mathrm{P}[N(5{,}000;1{,}000^2)>6{,}000]+(0.4)\mathrm{P}[N(4{,}000;1{,}000^2)>6{,}000]\\&=0.6(1-\Phi(1))+0.4(1-\Phi(2))=0.6(0.1587)+0.4(0.0228)\\&=\boxed{0.104}\end{aligned}$$

Note that the resulting mixture is not a normal random variable. For one thing, it is bi-modal, with the density having one local max at 4,000 and a second one at 5,000, while a normal random variable has a single mode.

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3. Suppose $X$ and $Y$ are independent normals, each with variance 1,000,000, and with $\mathrm{E}[X] = 5{,}000$ and $\mathrm{E}[Y] = 4{,}000$.

   Find $\mathrm{P}[0.6X + 0.4Y > 6{,}000]$

   A. 0.026 B. 0.052 C. 0.081 D. 0.104 E. 0.123

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Here we have a weighted average, which is a type of sum, so

$$\begin{aligned}
\mathrm{E}[0.6X + 0.4Y] &= 0.6 \cdot 5{,}000 + 0.4 \cdot 4{,}000 = 4{,}600 \\
\mathrm{Var}[0.6X + 0.4Y] &= 0.6^2 \mathrm{Var}[X] + 0.4^2 \mathrm{Var}[Y] \\
\mathrm{SD}[0.6X + 0.4Y] &= 721.11 \\
\mathrm{P}[0.6X + 0.4Y > 6{,}000] &= 1 - \Phi\left(\frac{6{,}000 - 4{,}600}{721.11}\right) = 1 - \Phi(1.94) \\
&= \boxed{0.0262}
\end{aligned}$$

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4. [1.S01.33] For Company A there is a 60% chance that no claim is made during the coming year. If one of more claims are made, the total claim amount is normally distributed with mean 10,000 and standard deviation 2,000.

   For Company B there is a 70% chance that no claim is made during the coming year. If one or more claims are made, the total claim amount is normally distributed with mean 9,000 and standard deviation 2,000.

   Assume that the total claim amounts of the two companies are independent.

   What is the probability that, in the coming year, Company B's total claim amount will exceed Company A's total claim amount?

   A. 0.180 B. 0.185 C. 0.217 D. 0.223 E. 0.240

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Neither $A$ nor $B$ is normally distributed because there is a positive probability of no claim being made. To use what we know about normal distributions, we have to condition on a payment being made for both $A$ and $B$.

I will use $A > 0$ and $B > 0$ to denote the case when $A$ and $B$ have claims (that is technically wrong as there is a very small probability of the normal variables being negative, but it simplifies the notation).

$$\begin{aligned}
\mathrm{P}[B > A] &= \mathrm{P}[B > A, A > 0] + \mathrm{P}[B > A, A = 0] \\
&= \mathrm{P}[A > 0] \cdot \mathrm{P}[B > A \mid A > 0] + \mathrm{P}[A = 0] \cdot \mathrm{P}[B > A \mid A = 0] \\
&= 0.4 \cdot \mathrm{P}[B > 0] \cdot \mathrm{P}[B > A \mid A, B > 0] + 0.6 \cdot \mathrm{P}[B > 0] \\
&= (0.4)(0.3)\mathrm{P}[B > A \mid A, B > 0] + (0.6)(0.3)
\end{aligned}$$

And when $A$ and $B$ both have claims we have $B - A$ is a normal random variable with parameters

$$\mathrm{E}[B - A \mid A, B > 0] = \mathrm{E}[B \mid B > 0] - \mathrm{E}[A \mid A > 0] = 9{,}000 - 10{,}000 = -1{,}000$$

$$
\begin{aligned}
\mathrm{Var}[B-A \mid A,B>0] &= \mathrm{Var}[B \mid B>0] + (-1)^2\mathrm{Var}[A \mid A>0] \\
&= 2{,}000^2 + 2{,}000^2 = 8{,}000{,}000 \\
\mathrm{SD}[B-A \mid A,B>0] &= 2{,}828 \\
\mathrm{P}[B-A>0 \mid A,B>0] &= \mathrm{P}\left[\frac{B-A-(-1{,}000)}{2{,}828} > \frac{0-(-1{,}000)}{2{,}828} \mid A,B>0\right] \\
&= 1-\Phi\left(\frac{1{,}000}{2{,}828}\right) \\
&= 1-\Phi(0.35) = 1-0.64 = 0.36 \\
\mathrm{P}[B>A] &= 0.4\cdot 0.3\cdot 0.36 + 0.6\cdot 0.3 = \boxed{0.223}
\end{aligned}
$$

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5. Loss amounts are normally distributed, with a 6.68% chance of exceeding 102 and a 2.28% chance of exceeding 105. What is the probability that a randomly selected loss is greater than 95?

A. 0.11 B. 0.37 C. 0.48 D. 0.66 E. 0.91

$1-6.68\% = 0.9332$, and looking that up on the normal table that is a z-value of 1.5. So 102 is 1.5 standard deviations above the mean. $1-0.0228 = 0.9772$ which has a z-value of 2, so 105 is 2 standard deviations above the mean. That means that

$$
\begin{aligned}
105-102 &= 0.5\mathrm{SD}[X] \\
\mathrm{SD}[X] &= 6 \\
\mathrm{E}[X] &= 105 - 2\cdot 6 = 93 \\
\mathrm{P}[X>95] &= \mathrm{P}\left[\frac{X-\mathrm{E}[X]}{\mathrm{SD}[X]} > \frac{95-93}{6}\right] \\
&= 1-\Phi(0.33) = 1-0.63 = \boxed{0.37}
\end{aligned}
$$

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6. Let $Y$ be a mixture of $X_1$ and $X_2$, where $X_1$ is a normal random variable with mean 0 and standard deviation 1, and $X_2$ is a normal random variable with mean 0 and standard deviation 5. If $\mathrm{P}[Y=X_1] = 0.9$, what is the kurtosis of $Y$?

Recall that the kurtosis of a normal random variable is 3.

A. 3.0 B. 8.2 C. 12.4 D. 16.5 E. 49.5

$\mathrm{Kurtosis}(X) = \dfrac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4}$ and so since the kurtosis of a normal is 3, $\mathrm{E}[Z^4] = 3$ if $Z$ is a standard normal.

$$
\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[\mathrm{E}[Y \mid \mathrm{Case}]] \\
&= \mathrm{P}[Y=X_1]\cdot\mathrm{E}[X_1] + \mathrm{P}[Y=X_2]\cdot\mathrm{E}[X_2] = 0.9\cdot 0 + 0.1\cdot 0 = 0 \\
\mathrm{Var}[Y] &= \mathrm{E}[\mathrm{Var}[Y \mid \mathrm{Case}]] + \mathrm{Var}[\mathrm{E}[Y \mid \mathrm{Case}]]
\end{aligned}
$$

$$
\begin{aligned}
&= 0.9 \cdot \mathrm{Var}[X_1] + 0.1 \cdot \mathrm{Var}[X_2] + 0.9 \cdot 0.1 \cdot (0-0)^2 \\
\mathrm{Var}[Y] &= 3.4 \\
\text{Or: } \mathrm{E}[Y^2] &= \mathrm{E}[\mathrm{E}[Y^2 \mid \text{Case}]] \\
&= \mathrm{P}[Y = X_1] \cdot \mathrm{E}[X_1^2] + \mathrm{P}[Y = X_2] \cdot \mathrm{E}[X_2^2] \\
&= 0.9 \cdot (0^2 + 1^2) + 0.1 \cdot (0^2 + 5^2) = 3.4 \\
\mathrm{Var}[Y] &= 3.4 - 0^2 = 3.4 \\
\mathrm{E}\left[(Y - \mathrm{E}[Y])^4\right] = \mathrm{E}\left[Y^4\right] &= 0.9 \cdot \mathrm{E}\left[X_1^4\right] + 0.1 \cdot \mathrm{E}\left[X_2^4\right] \\
&= 0.9 \cdot 3 + 0.1 \cdot 3 \cdot 5^4 \\
\mathrm{E}\left[Y^4\right] &= 190.2 \\
\mathrm{Kurtosis}(Y) &= \frac{190.2}{3.4^{4/2}} = \boxed{16.45}
\end{aligned}
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7. Losses are lognormal with median 3 and mean 4. What is the variance of a randomly selected loss?

A. 0.6 B. 2.7 C. 4.5 D. 7.4 E. 12.4

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The median of the underlying normal is $\mu$, so the median of the lognormal is $e^\mu$. Finding the other moments in the tables,

$$
\begin{aligned}
\text{Median} &= e^\mu = 3 \\
\mathrm{E}[X] &= e^{\mu + \sigma^2/2} = 4 \\
e^{\sigma^2/2} &= \frac{4}{3} \\
\mathrm{E}[X^2] &= e^{2\mu + 2\sigma^2} = (e^\mu)^2 \cdot (e^{\sigma^2/2})^4 \\
&= 9 \cdot \left(\frac{4}{3}\right)^4 = \frac{256}{9} \\
\mathrm{Var}[X] &= \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \\
&= \frac{256}{9} - 16 = \frac{112}{9} = \boxed{12.4}
\end{aligned}
$$

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8. Losses are lognormal with mean 3 and standard deviation 2. What is the probability that a loss that exceeds 1 will be greater than 4?

A. 0.23 B. 0.26 C. 0.37 D. 0.86 E. 0.89

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$$
\begin{aligned}
\mathrm{E}[X] &= 3 = e^{\mu + \sigma^2/2} \\
\mathrm{E}[X^2] &= 3^2 + 2^2 = e^{2\mu + 2\sigma^2} \\
2\ln(3) &= 2\mu + \sigma^2 \\
\ln(13) &= 2\mu + 2\sigma^2
\end{aligned}
$$

$$\sigma^2 = 0.3677$$
$$\mu = 0.915$$
$$\mathrm{P}[X > 4 \mid X > 1] = \frac{\mathrm{P}[X > 4]}{\mathrm{P}[X > 1]}$$
$$\mathrm{P}[X > 4] = \mathrm{P}[\ln(X) > \ln(4)] = 1 - \Phi\left(\frac{\ln(4) - \mu}{\sigma}\right)$$
$$= 1 - \Phi(0.78) = 0.2177$$
$$\mathrm{P}[X > 1] = \mathrm{P}[\ln(X) > \ln(1)] = 1 - \Phi\left(\frac{\ln(1) - \mu}{\sigma}\right)$$
$$= \Phi(1.51) = 0.9345$$
$$\mathrm{P}[X > 4 \mid X > 1] = \frac{0.2177}{0.9345} = \boxed{0.233}$$

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9. [4B.F97.26] You are given the following:

- In 1996, losses follow a lognormal distribution with parameters $\mu$ and $\sigma$.
- In 1997, losses follow a lognormal distribution with parameters $\mu + \ln(k)$ and $\sigma$, where $k > 1$.
- In 1996, $100p\%$ of the losses exceed the mean of the losses in 1997.

Determine $\sigma$. Note: $z_p$ is the $100p$th percentile of a normal distribution with mean 0 and variance 1.

A. $2\ln(k)$ B. $-z_p \pm \sqrt{z_p^2 - 2\ln(k)}$ C. $z_p \pm \sqrt{z_p^2 - 2\ln(k)}$ D. $\sqrt{-z_p \pm \sqrt{z_p^2 - 2\ln(k)}}$ E. $\sqrt{z_p \pm \sqrt{z_p^2 - 2\ln(k)}}$

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Let $X$ denote a 1996 loss amount.

$$\mathrm{P}\left[X > e^{\mu + \ln(k) + \sigma^2/2}\right] = p$$
$$\mathrm{P}\left[X \le e^{\mu + \ln(k) + \sigma^2/2}\right] = 1 - p$$
$$\mathrm{P}\left[\frac{\ln(X) - \mu}{\sigma} \le \frac{\ln(k) + \sigma^2/2}{\sigma}\right] = 1 - p$$
$$z_{1-p} = -z_p = \frac{\ln(k) + \sigma^2/2}{\sigma}$$
$$\sigma^2 + 2z_p\sigma + 2\ln(k) = 0$$
$$\sigma = \frac{-2z_p \pm \sqrt{4z_p^2 - 4 \cdot 2\ln(k)}}{2}$$
$$= \boxed{-z_p \pm \sqrt{z_p^2 - 2\ln(k)}}$$

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10. Loss amounts have a survival function given by

$$S(x) = \begin{cases} 1 & x < 0 \\ e^{-2x^2} & x \geq 0 \end{cases}$$

What is the average loss amount?

A. $\sqrt{\frac{\pi}{8}}$  B. $\sqrt{\frac{\pi}{4}}$  C. $\sqrt{\frac{\pi}{2}}$  D. $\sqrt{\pi}$  E. $\sqrt{2\pi}$

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A standard normal has density $f(x) = e^{-x^2/2}/\sqrt{2\pi}$, and a $N(\mu, \sigma^2)$ has density $\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$, so one way to integrate $e^{-x^2}$ is to compare with a normal CDF.

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^\infty e^{-2x^2}\,dx \\
&= \sigma\sqrt{2\pi} \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/(2\sigma^2)}\,dx \quad \text{for } 2 = 1/(2\sigma^2) \text{ and } \sigma = 1/2 \\
&= \frac{1}{2}\sqrt{2\pi} \cdot \mathrm{P}[Z > 0] \quad \text{for } Z \sim N(0, 1/4) \\
&= \frac{1}{2}\sqrt{2\pi} \cdot \frac{1}{2} = \boxed{\sqrt{\frac{\pi}{8}}}
\end{aligned}$$

Or: $X$ is a Weibull with $\tau = 2$ and $(1/\theta)^\tau = (1/\theta)^2 = 2$ so $\theta = 1/\sqrt{2}$ and $\mathrm{E}[X] = \frac{1}{\sqrt{2}}\Gamma(3/2) = \frac{1}{\sqrt{2}}\frac{1}{2}\sqrt{\pi} = \sqrt{\frac{\pi}{8}}$

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11. Loss amounts have a survival function given by

$$S(x) = \begin{cases} 1 & x < 1 \\ e^{-2x^2} & x \geq 1 \end{cases}$$

What is the average loss amount?

A. 0.03  B. 0.77  C. 1.03  D. 1.77  E. 2.41

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Now we have

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^1 1dx + \int_1^\infty e^{-2x^2}\,dx \\
&= 1 + \sqrt{\frac{\pi}{2}}\mathrm{P}[Z > 1] \\
&= 1 + \sqrt{\frac{\pi}{2}}\mathrm{P}\left[\frac{Z-0}{1/2} > \frac{1-0}{1/2}\right]
\end{aligned}$$

$$= 1 + \sqrt{\frac{\pi}{2}}(1 - \Phi(2)) = \boxed{1.029}$$

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12. The density function of a random variable is proportional to $e^{-x^2}$ for $x \geq 0.5$ and is 0 otherwise. Find $\mathrm{P}[X \geq 1]$

A. 0.08 B. 0.33 C. 0.55 D. 0.76 E. 0.92

$f(x) = ce^{-x^2}$ for $x > 0.5$, where $c = 1/\int_{0.5}^{\infty} e^{-x^2}dx$ so

$$\begin{aligned}
\mathrm{P}[X \geq 1] &= \frac{\int_1^\infty e^{-x^2}dx}{\int_{0.5}^\infty e^{-x^2}dx} \\
&= \frac{\mathrm{P}[Z > 1]}{\mathrm{P}[Z > 0.5]} \text{ where } Z \text{ is a normal with mean 0 and variance } 1/2 \\
&= \frac{1 - \Phi\left(\frac{1}{1/\sqrt{2}}\right)}{1 - \Phi\left(\frac{0.5}{1/\sqrt{2}}\right)} \\
&= \frac{1 - 0.9207}{1 - 0.7611} = \boxed{0.33}
\end{aligned}$$

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13. Losses are modeled with a lognormal distribution with parameters $\mu$ and $\sigma$. If the median loss amount is 1.06 and the mean loss amount is 1.08, what is $\sigma$?

A. 0.01 B. 0.02 C. 0.04 D. 0.10 E. 0.19

For a lognormal, the median is $e^\mu$ and the mean is $e^{\mu+\sigma^2/2}$ so

$$\begin{aligned}
e^\mu &= 1.06 \\
\mu &= \ln 1.06 = 0.0583 \\
e^{\mu+\sigma^2/2} &= 1.08 \\
\mu + \sigma^2/2 &= \ln 1.06 + \sigma^2/2 = \ln 1.08 \\
\sigma &= \boxed{0.1933}
\end{aligned}$$

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14. Losses are modeled with a lognormal distribution with mean 0.42 and variance 1.65. Find the probability that losses are at least 1.

A. 0.09 B. 0.18 C. 0.27 D. 0.33 E. 0.36

$$e^{\mu+\sigma^2/2} = 0.42 \qquad e^{2\mu+\sigma^2} = 0.42^2$$

$$
\begin{aligned}
e^{2\mu+2\sigma^2} &= 1.65 + 0.42^2 = 1.8264 \\
e^{\sigma^2} &= \frac{1.8264}{0.42^2} \quad \text{by division} \\
\sigma &= 1.5288 \\
\ln 0.42 &= \mu + 1.5288^2/2, \quad \mu = -2.036 \\
\mathrm{P}[X > 1] &= \mathrm{P}[\ln X > 0] \\
&= \mathrm{P}\left[\frac{\ln X - \mu}{\sigma} > \frac{2.036}{1.5288}\right] = 1 - \Phi(1.33) \\
&= \boxed{0.0918}
\end{aligned}
$$

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15. Suppose that $X$ and $Y$ are jointly normal random variables, with $\mathrm{E}[X] = 1$, $\mathrm{Var}[X] = 4$, and $\mathrm{E}[Y] = -2$, $\mathrm{E}\left(Y^2\right) = 5$. If the correlation of $X$ and $Y$ is $-1/2$, what is the probability that the sum of $X$ and $Y$ is positive?

A. 0.16 B. 0.28 C. 0.37 D. 0.72 E. 0.84

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Since $X$ and $Y$ are jointly normal, $X+Y$ is also a normal random variable. $\mathrm{E}[X+Y] = \mathrm{E}[X]+\mathrm{E}[Y] = -1$ and

$$
\begin{aligned}
\mathrm{Var}[X+Y] &= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}(X,Y) \\
&= 4 + (5 - (-2)^2) + 2\mathrm{Corr}(X,Y)\mathrm{SD}[X]\mathrm{SD}[Y] \\
&= 4 + 1 + 2(-1/2)(2)(1) = 3
\end{aligned}
$$

Combining those, we obtain

$$
\mathrm{P}[X+Y>0] = \mathrm{P}\left[\frac{X+Y-(-1)}{\sqrt{3}} > \frac{0-(-1)}{\sqrt{3}}\right] = 1 - \Phi(0.58) = 1 - 0.72 = \boxed{0.28}
$$

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16. Variant on [3-CAS.F05.32] Seventy-five percent of claims have a normal distribution with a mean of 3,000 and a variance of 1,000,000. The remaining 25% have a normal distribution with a mean of 4,000 and a variance of 1,000,000. Determine the probability that a randomly chosen claim exceeds 5,000.

A. Less than 0.040
B. At least 0.040, but less than 0.045
C. At least 0.045, but less than 0.050
D. At least 0.050, but less than 0.055
E. At least 0.055

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Let Type I denote the claims with a mean of 3,000, and Type II those claims with a mean of 4,000. To simplify the computations, it is easier work in units of 1,000. Then we want $\mathrm{P}[X > 5]$, which is

$$
\mathrm{P}[X>5] = \mathrm{P}[\text{Type I}]\mathrm{P}[X>5 \mid \text{Type I}] + \mathrm{P}[\text{Type II}]\mathrm{P}[X>5 \mid \text{Type II}]
$$

$$
\begin{aligned}
&= 0.75\left[1-\Phi\left(\frac{5-3}{1}\right)\right] + 0.25\left[1-\Phi\left(\frac{5-4}{1}\right)\right] \\
&= 0.75(1-\Phi(2)) + 0.25(1-\Phi(1)) \\
&= 0.057
\end{aligned}
$$

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17. Loss amounts $X$ have a lognormal distribution with parameters $\mu = 2$ and $\sigma^2 = 0.64$. What is the skewness of $X$?

A. 0.4 B. 3.7 C. 6.8 D. 10.2 E. 14.1

$$
\begin{aligned}
\mathrm{E}\big[(X-\mu_X)^3\big] &= \mathrm{E}\big[X^3\big] - 3\mathrm{E}\big[X^2\big]\,\mu_X + 2\mu_X^3 \\
&= e^{3\cdot 2+(3^2/2)\cdot 0.64} - 3\cdot e^{2\cdot 2+2\cdot 0.64}\cdot e^{2+0.5\cdot 0.64} + 2e^{(2+0.5\cdot 0.64)\cdot 3} \\
&= 3{,}299.47 \\
\mathrm{Var}(X) &= e^{5.28} - e^{2\cdot 2.32} \\
\mathrm{SD}(X) &= 9.6346 \\
\mathrm{Skew}(X) &= \frac{3{,}300}{9.6346^3} = \boxed{3.7}
\end{aligned}
$$

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