A.2 Key Distributions - Outline

A.2.4 Normal and Lognormal Distributions
- Normals
- Lognormals
- Exercises

Normal

A standard normal $Z \sim N(0,1)$ has cdf $\Phi(z)$ and density

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \quad \text{(on tables)}$$

$$\int_0^\infty \phi(z)\, dz = 1 - \Phi(0) = \frac{1}{2}$$

$$\int_0^\infty e^{-z^2/2}\, dz = \frac{\sqrt{2\pi}}{2}$$

$\Phi(z)$ is given on tables for $z > 0$.
For $z < 0$, use $\Phi(z) = 1 - \Phi(-z)$.
If $t < 0.5$, then $\Phi^{-1}(t) = -\Phi^{-1}(1-t)$.

When looking up $\Phi(z)$ or $\Phi^{-1}(z)$, round to the nearest value on the table. *Don't interpolate!*
E.g., $\Phi(-1.234) = 1 - \Phi(1.234) = 1 - \Phi(1.23) = 0.1093$.

Sums and Transformations

If $X \sim N(\mu, \sigma^2)$ then $X = \sigma Z + \mu$ where $Z \sim N(0, 1)$

$$\mathrm{P}[X \leq x] = \mathrm{P}\left[\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right] = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

Linear combinations (e.g., sums and weighted averages) of independent (or just bivariate) normals are normal. Mixtures are not.

Example

If $X \sim N(10, 50)$, and $Y \sim N(20, 60)$ are independent, find $\mathrm{P}[X + Y > 40]$.

$$\begin{aligned} \mathrm{E}[X+Y] &= \mathrm{E}[X] + \mathrm{E}[Y] = 30 \\ \mathrm{Var}[X+Y] &= \mathrm{Var}[X] + \mathrm{Var}[Y] = 110 \\ \mathrm{P}[X+Y > 40] &= 1 - \Phi\left(\frac{40-30}{\sqrt{110}}\right) \\ &\approx 1 - \Phi(0.95) = \boxed{0.1711} \end{aligned}$$

Mixture

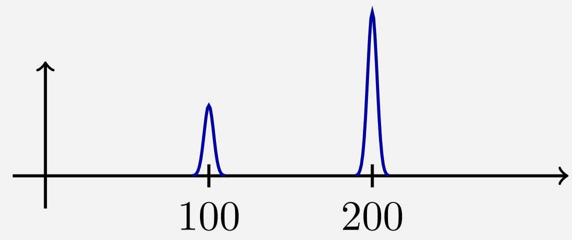
Losses from low risk individuals are $N(100, 1)$, while losses from high risk individuals are $N(200, 1)$. If 30% of losses are from low risk individuals, what is the probability that a randomly selected loss exceeds 200?
What is the mean and variance of a randomly selected loss?

$$\begin{aligned} \mathrm{P}[X > 200] &= 0.3 \cdot \mathrm{P}[N(100,1) > 200] + 0.7 \cdot \mathrm{P}[N(200,1) > 200] \\ &\approx 0.3 \cdot 0 + 0.7 \cdot 0.5 = 0.35 \\ \mathrm{E}[X] &= 0.3 \cdot 100 + 0.7 \cdot 200 = 170 \\ \mathrm{Var}[X] &= \mathrm{E}[\mathrm{Var}[X \mid \mathrm{Case}]] + \mathrm{Var}[\mathrm{E}[X \mid \mathrm{Case}]] \\ &= 0.3 \cdot 1 + 0.7 \cdot 1 + (200-100)^2 \cdot 0.3 \cdot 0.7 = 2{,}101 \end{aligned}$$

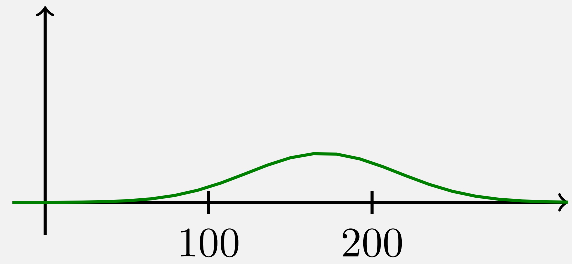
Note: A normal has a single mode at the mean. But instead of having a single mode at 170, our distribution is bimodal.
Bottom line: *Mixtures of normals are not normal.*

Mixture Example Continued

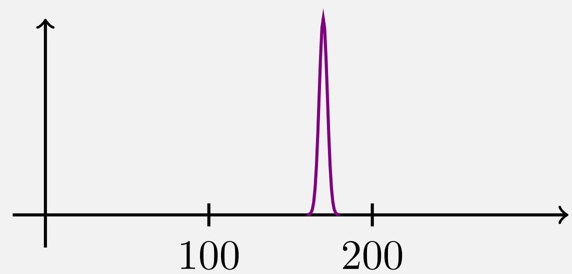
That was a two-point mixture:
$X \sim N(100, 1)$ 30% of the time,
$X \sim N(200, 1)$ 70% of the time.
The graph of $f(x)$ is



$\mathrm{E}[X] = 170, \mathrm{Var}[X] = 2{,}101$.
If $Y \sim N(170, 2{,}101)$, then $f(y)$ looks like



If $Z_1 \sim N(100, 1), Z_2 \sim N(200, 1)$ are independent.
$W = 0.3Z_1 + 0.7Z_2$
W is a sum so is normal
$\mathrm{E}[W] = 170, \mathrm{Var}[W] = 0.58$
$f(w)$ looks like



Lognormal

Suppose $Y \sim N(\mu, \sigma^2)$ and $X = e^Y$. Then X is a lognormal. The tables use the same μ and σ^2 as the parameters of X as well.

For moments of X we use the tables. If $\mu = 2$ and $\sigma = 1.4$ then

$$\mathrm{E}[X] = e^{\mu+\sigma^2/2} = e^{2+1.4^2/2} = 19.7$$

$$\mathrm{E}[X^2] = e^{2\mu+2^2\sigma^2/2} = e^{4+2\cdot 1.4^2} = 2{,}752$$

Note: μ and σ^2 are the mean and variance of the underlying normal Y and not of the lognormal X.

For probabilities, we take logs and work with the normal.

$$\begin{aligned} \mathrm{P}[X > 19.7] &= \mathrm{P}[Y > \ln(19.7)] \\ &= 1 - \mathrm{P}\left[\frac{Y-2}{1.4} \leq \frac{\ln(19.7)-2}{1.4}\right] \\ &= 1 - \Phi(0.7) = 0.242 \end{aligned}$$

Exercise 1

X is lognormal with mean 10 and variance 200. Find $\mathrm{P}[X > 10]$.

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X is lognormal with mean 10 and variance 200. Find $\mathrm{P}[X > 10]$.

$$
\begin{aligned}
\mathrm{E}[X] &= 10 = e^{\mu+\sigma^2/2} \\
\mathrm{E}[X^2] &= 10^2 + 200 = e^{2\mu+2\sigma^2} \\
2\mu + \sigma^2 &= 2\ln(10) \\
2\mu + 2\sigma^2 &= \ln(300) \\
\sigma^2 &= \ln(300) - 2\ln(10) = 1.099 \\
\mu &= 1.753 \\
\mathrm{P}[X > 10] &= \mathrm{P}[\ln(X) > \ln(10)] \\
&= \mathrm{P}\left[\frac{\ln(X) - \mu}{\sigma} > \frac{\ln(10) - 1.753}{\sqrt{1.099}}\right] \\
&= 1 - \Phi(0.52) = \Phi(-0.52) = \boxed{0.3015}
\end{aligned}
$$

Warning: Lognormal problems are more sensitive to rounding errors than most problems. Carry as many digits as possible.

Exercise 2

Suppose that X is normal with $\mathrm{P}[X > 5] = 0.5$ and $\mathrm{P}[X > 8] = 0.05$. Find $\mathrm{E}[e^{2X}]$.

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$$
\begin{aligned}
\mu &= 5 \\
\frac{8-5}{\sigma} &= 1.645 \\
\sigma &= 1.8237 \\
e^X &\sim \mathrm{LN}(\mu = 5, \sigma^2 = 3.3259) \\
\mathrm{E}[e^{2X}] &= \mathrm{E}[(e^X)^2] \\
&= e^{2\mu + 2\sigma^2} \\
&= \boxed{17{,}052{,}771}
\end{aligned}
$$