A.2.1 Probability Functions

Future Lifetime of a newborn

Future Lifetime of (x)

Cumulative Distribution

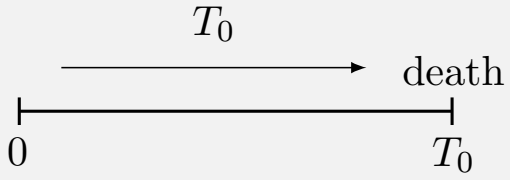
Survival Distribution

Curtate Future Lifetime

Exercises

Future Lifetime of a newborn, T_0

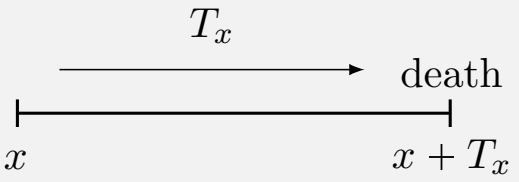
- $T_0 \sim$ RV for future lifetime for a newborn
- In other words, the age-at-death RV
- T_0 is continuous
- $T_0 \geqslant 0$



age-at-death RV $\equiv T_0$

Future Lifetime of a person age x, T_x

- (x) - a life age x
- $T_x \sim$ RV for future lifetime of (x)
- T_x is continuous
- $T_x \geqslant 0$



age-at-death RV $\equiv x + T_x$

$$T_0 = x + T_x$$

Cumulative Distribution Function, $F_x(t)$

Probability of (x) dying before age $x + t$.

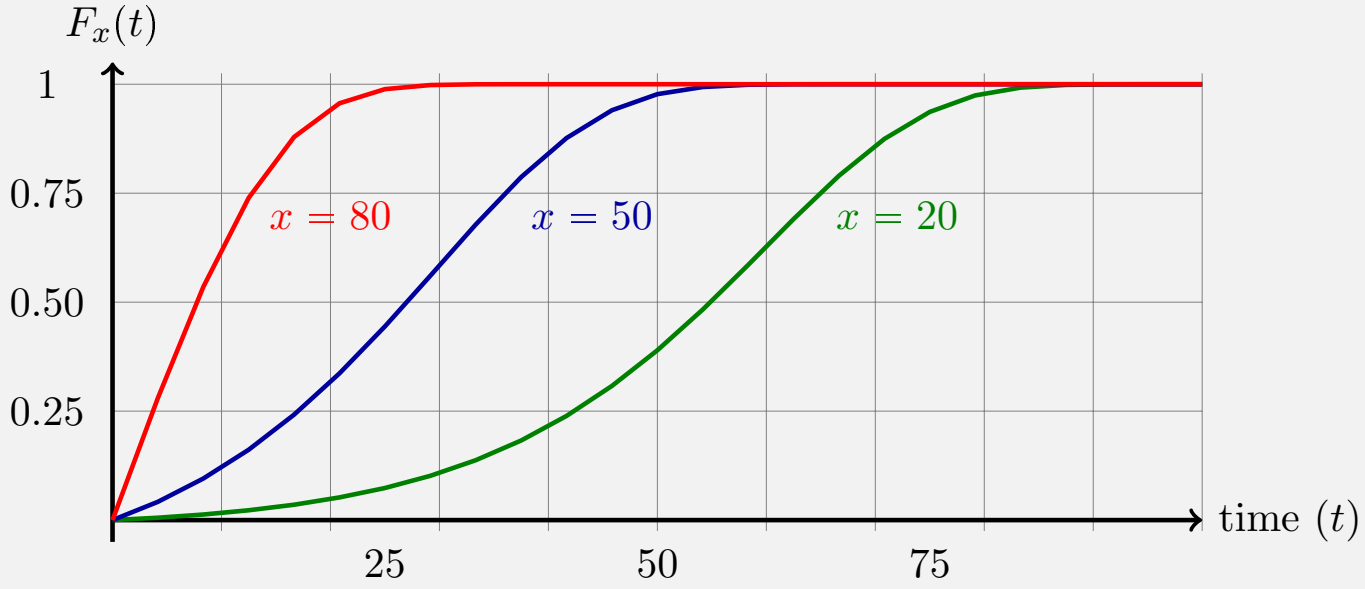
$$F_x(t) = \Pr(T_x \leqslant t) = \Pr(T_0 \leqslant x + t \,|\, T_0 > x) = \frac{F_0(x+t) - F_0(x)}{1 - F_0(x)}$$

Three properties:

1. $F_x(0) = 0$
2. $\lim_{t \to \infty} F_x(t) = 1$
3. non-decreasing function of t: $F_x(a) \leqslant F_x(b)$ for $a < b$

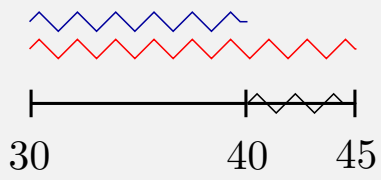
   $F_{30}(5) \not> F_{30}(50)$

Cumulative Distribution Function Example Graph



Cumulative Distribution Function Example

Express the probability that a (30) dies between ages 40 and 45 using the cumulative distribution function.



$$\begin{aligned}\Pr(10 < T_{30} < 15) &= F_{30}(15) - F_{30}(10) \\ &= \Pr(40 < T_0 < 45 \,|\, T_0 > 30) \\ &= \frac{F_0(45) - F_0(40)}{1 - F_0(30)}\end{aligned}$$

Survival Distribution Function, $S_x(t)$

Probability of a (x) attaining age $x+t$. i.e., probability of a (x) not dying before age $x+t$.

$$S_x(t) = \Pr(T_x > t) = 1 - \Pr(T_x \leqslant t) = 1 - F_x(t)$$

$$= \Pr(T_0 > x+t \,|\, T_0 > x) = \frac{S_0(x+t)}{S_0(x)}$$

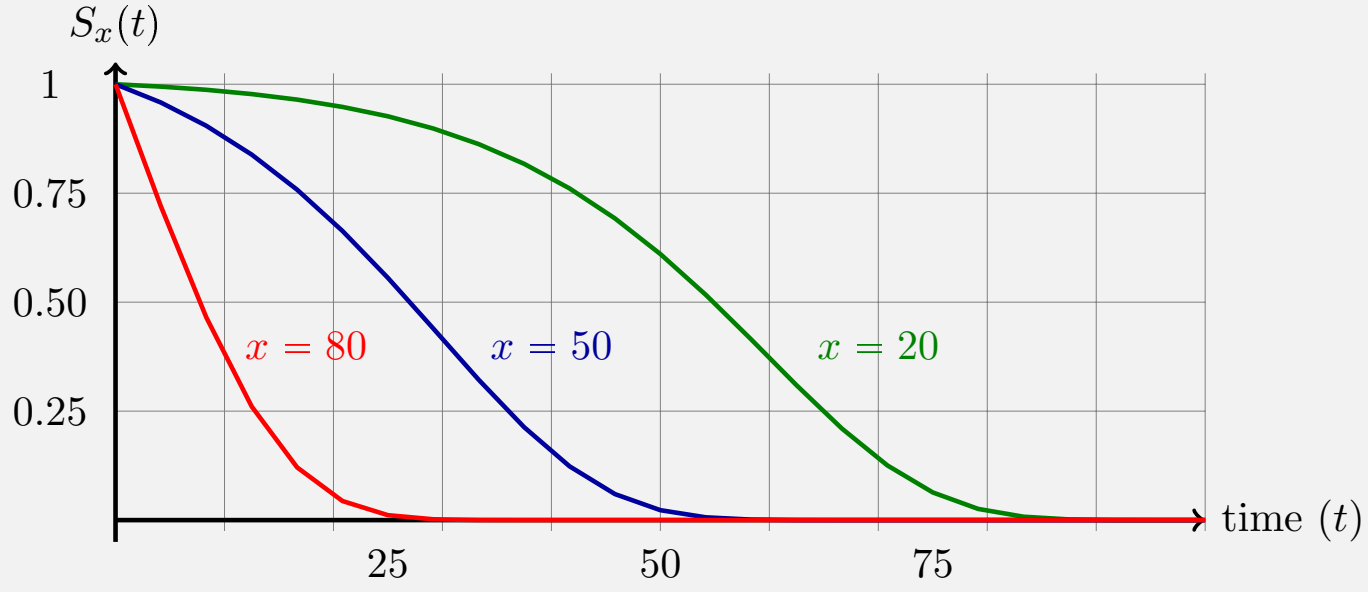
$$S_0(x+t) = S_0(x) \cdot S_x(t)$$

Three properties:

1. $S_x(0) = 1$
2. $\lim_{t \to \infty} S_x(t) = 0$
3. non-increasing function of t: $S_x(a) \geqslant S_x(b)$ for $a < b$

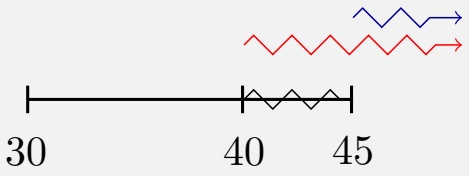
   $S_{30}(5) \nleqslant S_{30}(50)$

Survival Distribution Function Example Graph



Survival Distribution Example

Express the probability that a (30) dies between ages 40 and 45 using the survival function.



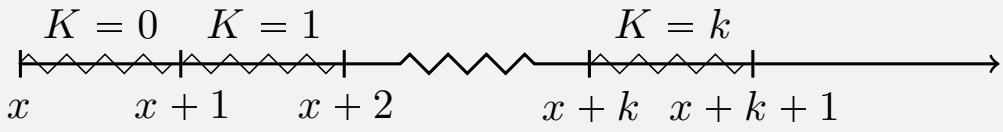
$$\begin{aligned}\Pr(10 < T_{30} < 15) &= S_{30}(10) - S_{30}(15) \\ &= \frac{S_0(40) - S_0(45)}{S_0(30)}\end{aligned}$$

Curtate Future Lifetime of (x), K_x

$$K_x = \lfloor T_x \rfloor$$

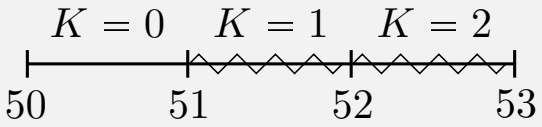
where $\lfloor\ \rfloor$ denotes the greatest integer (or floor) function

$K_x \sim$ RV for the number of completed future years by (x) prior to death. For integral x, K_x is the number of future birthdays.



Curtate Future Lifetime Example

Write the probability that a 50-year-old dies between ages 51 and 53 using the curtate future lifetime, K_x.



$$\Pr(1 \leqslant K_{50} \leqslant 2) = \Pr(K_{50} = 1) + \Pr(K_{50} = 2)$$

Exercise 1

Describe the following in words:

1. T_{20}
2. $F_0(40)$
3. $S_{20}(30)$
4. K_{50}

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Answers:

1. T_{20} - future lifetime random variable for (20)
2. $F_0(40)$ - probability a newborn dies within 40 years
3. $S_{20}(30)$ - probability (20) survives to age 50
4. K_{50} - curtate future lifetime random variable of (50)

Exercise 2

You are given the following survival function:

$$S_0(x) = \begin{cases} \dfrac{10000 - x^2}{10000} & 0 \leqslant x \leqslant 100 \\ 0 & x > 100 \end{cases}$$

Find the probability that (32) dies before age 33.

Exercise 2

You are given the following survival function:

$$S_0(x) = \begin{cases} \dfrac{10000 - x^2}{10000} & 0 \leqslant x \leqslant 100 \\ 0 & x > 100 \end{cases}$$

Find the probability that (32) dies before age 33.

$$\begin{aligned} F_{32}(1) &= \Pr[32 < T_0 < 33 | T_0 > 32] \\ &= \frac{S_0(32) - S_0(33)}{S_0(32)} \\ &= \frac{\frac{10000-32^2}{10000} - \frac{10000-33^2}{10000}}{\frac{10000-32^2}{10000}} = \boxed{0.007} \end{aligned}$$

Exercise 3

You are given a survival function $S_0(x) = 1 - 0.01x$ for $0 \leqslant x \leqslant 100$.

Determine the median future lifetime of a life aged 10.

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You are given a survival function $S_0(x) = 1 - 0.01x$ for $0 \leqslant x \leqslant 100$.

Determine the median future lifetime of a life aged 10.

Let m be the median.

$$\Pr[T_{10} \leqslant m] = 0.5$$

$$\Pr[T_{10} > m] = 1 - 0.5 = 0.5$$

$$S_{10}(m) = 0.5$$

$$\frac{S_0(m+10)}{S_0(10)} = 0.5$$

$$\frac{1 - 0.01(m+10)}{1 - 0.01(10)} = 0.5$$

$$m = \boxed{45}$$