A.3.1 Force of Mortality – Part 1

Define Force of Mortality, μ_x

Probability Density Function, $f_x(t)$

Derive ${}_np_x$ formula

Properties for μ_x

${}_np_x$ Formula Examples

Exercises

Define Force of Mortality, μ_x

By definition the force of mortality μ_x is

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr[T_0 \leqslant x + dx \,|T_0 > x]$$

For very small dx it follows that

$$\mu_x \, dx \approx \Pr[T_0 \leqslant x + dx \,|T_0 > x]$$

Thus, $\mu_x \, dx$ is the probability that a life who has attained age x dies before attaining age $x + dx$.

Force of Mortality vs Force of Interest

Interest Theory	Life Contingencies
$\delta_t = \dfrac{\frac{d}{dt}a(t)}{a(t)}$	$\mu_x = \dfrac{-\frac{d}{dx}S_0(x)}{S_0(x)}$

$\delta_t\,dt =$ amount of interest earned on \$1 b/w t and $t + dt$

$\mu_x\,dx =$ probability (x) dies b/w ages x and $x + dx$

Probability Density Function, $f_x(t)$

Probability density function for future lifetime of (x), i.e. PDF for T_x.

$$f_x(t) = F'_x(t) = -S'_x(t)$$

Two properties:

1. $f_x(t) \geqslant 0$
2. $\int_0^\infty f_x(t)\,dt = 1$

$$\mu_x = -\frac{S'_0(x)}{S_0(x)} = \frac{f_0(x)}{S_0(x)}$$

$$f_0(x) = S_0(x)\,\mu_x$$

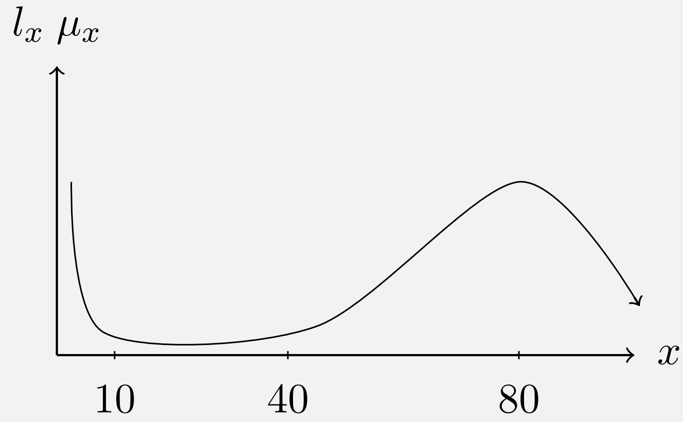
$$f_0(x)\,dx = S_0(x)\,\mu_x\,dx$$

Loosely: prob age at death is $x =$ prob survive to age x and then hit by force of mortality

Curve of Deaths

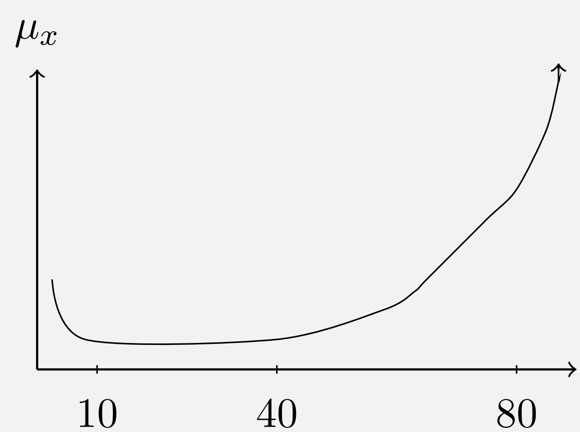
$$f_0(x) = S_0(x)\,\mu_x = \frac{l_x}{l_0}\cdot\mu_x \Rightarrow l_0\cdot f_0(x) = l_x\cdot\mu_x$$

$\therefore\ l_x\,\mu_x$ is the curve of deaths



Curve of Deaths

Force of Mortality Graph



Force of Mortality

Derive ${}_np_x$ formula

$$\mu_x = -\frac{S_0'(x)}{S_0(x)}$$

$$\mu_{x+t} = -\frac{S_0'(x+t)}{S_0(x+t)} = -\frac{d}{dt}\ln\, S_0(x+t)$$

$$-\int_0^n \mu_{x+t}\,dt = \ln\, S_0(x+n) - \ln\, S_0(x)$$

$$= \ln\left(\frac{S_0(x+n)}{S_0(x)}\right) = \ln\, {}_np_x$$

$${}_np_x = \exp\left[-\int_0^n \mu_{x+t}\,dt\right]$$

In words: add up the force of mortality between ages x and $x+n$, take the negative and take the exponential.

$S_0(n)$ vs $a(n)$

$$S_0(n) = {}_np_0 = \exp\left[-\int_0^n \mu_t\,dt\right]$$

$$a(n) = \exp\left[\int_0^n \delta_t\,dt\right]$$

Properties for μ_x

Two properties:

1. $\mu_x \geqslant 0$
2. $\int_0^\infty \mu_x \, dx = \infty$ because ${}_\infty p_0 = \exp\left(-\int_0^\infty \mu_x \, dx\right) = 0$

Revisit ${}_n p_x$ Formula

Recall

$$ {}_n p_x = \exp\left[-\int_0^n \mu_{x+t} \, dt\right] $$

Rewrite in terms of μ_y instead of μ_{x+t}

$$ y = x + t $$

$$ dy = dt $$

$$ {}_n p_x = \exp\left[-\int_x^{x+n} \mu_y \, dy\right] $$

In words: add up the force of mortality between ages x and $x+n$, take the negative and take the exponential. Same as before.

${}_np_x$ Formula Examples

Examples

$$\exp\left(-\int_0^{15} \mu_y \, dy\right) = {}_{15}p_0$$

$$\exp\left(-\int_0^{15} \mu_{20+y} \, dy\right) = {}_{15}p_{20}$$

$$\exp\left(-\int_5^{25} \mu_y \, dy\right) = {}_{20}p_5$$

Exercise 1

Write a formula for ${}_{10}p_{30}$ in terms of

A. μ_x

B. μ_{30+t}

C. μ_{15+t}

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Write a formula for ${}_{10}p_{30}$ in terms of

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B. μ_{30+t}

C. μ_{15+t}

A. $${}_{10}p_{30} = \exp\left(-\int_{30}^{40} \mu_x \, dx\right)$$

B. $${}_{10}p_{30} = \exp\left(-\int_{0}^{10} \mu_{30+t} \, dt\right)$$

C. $${}_{10}p_{30} = \exp\left(-\int_{15}^{25} \mu_{15+t} \, dt\right)$$

Exercise 2

You are given:

(i) ${}_{3}p_{70} = 0.95$

(ii) ${}_{2}p_{71} = 0.96$

(iii) $\int_{71}^{75} \mu_x \, dx = 0.107$

Calculate ${}_{5}p_{70}$.

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You are given:

(i) ${}_3p_{70} = 0.95$

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Calculate ${}_5p_{70}$.

Step 1 - Find ${}_4p_{71}$

$$
\begin{aligned}
{}_4p_{71} &= \exp\left[-\int_{71}^{75} \mu_x \, dx\right] \\
&= \exp\left[-0.107\right]
\end{aligned}
$$

Step 2 - Find p_{70}

$$
\begin{aligned}
{}_3p_{70} &= p_{70} \cdot {}_2p_{71} \\
0.95 &= p_{70} \cdot 0.96 \\
p_{70} &= \frac{95}{96}
\end{aligned}
$$

Step 3 - Find ${}_5p_{70}$

$$
\begin{aligned}
{}_5p_{70} &= p_{70} \cdot {}_4p_{71} \\
&= \frac{95}{96} \exp\left[-0.107\right] \\
&= \boxed{0.889}
\end{aligned}
$$

Exercise 3

Given that $S_0(x) = \exp(-\frac{x^3}{12})$, show that $\mu_x = \frac{x^2}{4}$.

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$$\begin{aligned}\mu_x = & -\frac{S_0'(x)}{S_0(x)} \\ = & -\frac{\exp(-\frac{x^3}{12})(-\frac{3x^2}{12})}{\exp(-\frac{x^3}{12})} \\ = & \frac{x^2}{4} \quad \checkmark\end{aligned}$$