A.4.1 Future Lifetime Moments - Discrete Case

Curtate Expectation of Life

Variance Curtate Future Lifetime

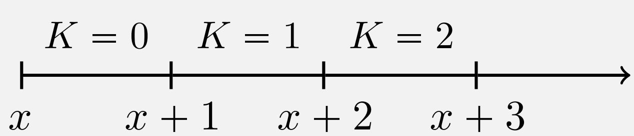
Recursion

Exercises

Curtate Future Lifetime

Recall

$$K_x = \text{number of whole years of future life for } (x)$$



$$\Pr[K_x = k] = {}_kp_x \cdot q_{x+k} = {}_{k|}q_x$$

$$\mathrm{E}[K_x] = \sum_{k=0}^{\infty} k \cdot {}_kp_x \cdot q_{x+k} = \sum_{k=0}^{\infty} k \cdot {}_{k|}q_x$$

Curtate Expectation of Life

$$\mathrm{E}[K_x] = \sum_{k=0}^{\infty} k \cdot {}_{k|}q_x = \sum_{k=0}^{\infty} k({}_kp_x - {}_{k+1}p_x)$$

$$\begin{aligned}\mathrm{E}[K_x] &= 1(p_x - {}_2p_x) + 2({}_2p_x - {}_3p_x) + 3({}_3p_x - {}_4p_x) + \cdots \\ &= p_x + {}_2p_x + {}_3p_x + {}_4p_x + \ldots \\ &= \sum_{k=1}^{\infty} {}_kp_x\end{aligned}$$

$$\mathrm{E}[K_x] = e_x = \sum_{k=1}^{\infty} {}_kp_x$$

- curtate expectation of life for (x)
- for integral x, expected number of future birthdays for (x)

Curtate Expectation of Life cont.

$$\begin{aligned}e_x &= p_x + {}_2p_x + {}_3p_x + \cdots \\ &= \frac{l_{x+1}}{l_x} + \frac{l_{x+2}}{l_x} + \frac{l_{x+3}}{l_x} + \cdots \\ &= \frac{l_{x+1} + l_{x+2} + l_{x+3} + \cdots}{l_x}\end{aligned}$$

average number of future birthdays

Life Table Example

x	l_x
0	81
1	64
2	49
3	36
4	25
5	16
6	9
7	4
8	1
9	0

For $l_0 = 81$, what is total # of FB?

$$64 + 49 + 36 + \cdots + 1 = 204$$

For $l_0 = 81$, what is average # of FB?

$$\frac{204}{81} = e_0$$

$$e_5 = \frac{9+4+1}{16} = \frac{14}{16}$$

Variance of Curtate Future Lifetime

$$\begin{aligned}
\mathrm{E}\left[K_x^2\right] &= \sum_{k=0}^{\infty} k^2 ({}_k p_x - {}_{k+1}p_x) \\
&= 1\,(p_x - {}_2p_x) + 4({}_2p_x - {}_3p_x) + 9({}_3p_x - {}_4p_x) \\
&\quad + 16({}_4p_x - {}_5p_x) + \cdots \\
&= p_x + 3\,{}_2p_x + 5\,{}_3p_x + 7\,{}_4p_x + \cdots \\
&= \sum_{k=1}^{\infty} (2k-1)\,{}_kp_x
\end{aligned}$$

$$\mathrm{Var}[K_x] = \mathrm{E}\left[K_x^2\right] - (e_x)^2$$

Curtate Future Lifetime Recursion

$$e_x = \frac{l_{x+1} + l_{x+2} + l_{x+3} + \cdots}{l_x} \cdot \frac{l_{x+1}}{l_{x+1}}$$

$$= \frac{l_{x+1}}{l_x} \cdot \frac{l_{x+1} + l_{x+2} + l_{x+3} + \cdots}{l_{x+1}}$$

$$e_x = p_x[1 + e_{x+1}]$$

$$e_x = p_x + {}_2p_x(1 + e_{x+2})$$

$$e_x = p_x + {}_2p_x + {}_3p_x(1 + e_{x+3})$$

Exercise 1

For a life aged 50, the curtate-expectation of life $e_{50} = 20$. For that same life, you are also given that $p_{50} = 0.97$.

Determine e_{51}.

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Determine e_{51}.

$$e_{50} = p_{50}\,(1 + e_{51})$$

$$20 = 0.97\,(1 + e_{51})$$

$$e_{51} = \boxed{19.619}$$

Exercise 2

You are given:

x	l_x
0	81
1	64
2	49
3	36
4	25
5	16
6	9
7	4
8	1
9	0

Calculate $\mathrm{Var}[K_6]$.

Exercise 2

You are given:

x	l_x
0	81
1	64
2	49
3	36
4	25
5	16
6	9
7	4
8	1
9	0

Calculate $\text{Var}[K_6]$.

$$\begin{aligned} \text{E}[K_6] &= p_6 + {}_2p_6 + {}_3p_6 \\ &= \tfrac{4}{9} + \tfrac{1}{9} + \tfrac{0}{9} \\ &= \tfrac{5}{9} \end{aligned}$$

$$\begin{aligned} \text{E}[K_6^2] &= (2\cdot 1 - 1)p_6 + (2\cdot 2 - 1)_2p_6 + (2\cdot 3 - 1)_3p_6 \\ &= \tfrac{4}{9} + 3(\tfrac{1}{9}) + 5(\tfrac{0}{9}) \\ &= \tfrac{7}{9} \end{aligned}$$

$$\text{Var}[K_6] = \tfrac{7}{9} - (\tfrac{5}{9})^2 = \boxed{0.469}$$

Exercise 2 - Alternate Solution

You are given:

x	l_x
0	81
1	64
2	49
3	36
4	25
5	16
6	9
7	4
8	1
9	0

Calculate $\text{Var}[K_6]$.

K_6	Probability
0	$\frac{9-4}{9} = \frac{5}{9}$
1	$\frac{4-1}{9} = \frac{3}{9}$
2	$\frac{1-0}{9} = \frac{1}{9} = 1 - \left(\frac{5}{9} + \frac{3}{9}\right)$

$$\text{E}[K_6] = 0(\tfrac{5}{9}) + 1(\tfrac{3}{9}) + 2(\tfrac{1}{9}) = \tfrac{5}{9}$$

$$\text{E}[K_6^2] = 0^2(\tfrac{5}{9}) + 1^2(\tfrac{3}{9}) + 2^2(\tfrac{1}{9}) = \tfrac{7}{9}$$

$$\text{Var}[K_6] = \tfrac{7}{9} - \left(\tfrac{5}{9}\right)^2 = \boxed{0.469}$$

Or use TI-30XS MultiView