4. Interest Theory Review

Converting between Rates

Accumulating and Discounting

Doubling the Force of Interest

Annuities

Integrals

Convert between i, d and δ

$$\begin{array}{c} d \xrightarrow{(1)} i \\ 0 \xleftarrow[(2)]{} 1 \end{array}$$

$$\frac{d}{1-d} = i \qquad (1)$$

$$\frac{i}{1+i} = d \qquad (2)$$

$$\delta = \ln(1+i)$$

Example

Given $i = 10\%$ find d and δ.

$$d = 0.09091$$

$$\delta = 0.09531$$

Accumulating and Discounting

Accumulating	Discounting
$1+i=e^{\delta}$	$v=(1+i)^{-1}=e^{-\delta}$
$(1+i)^2=e^{2\delta}$	$v^2=e^{-2\delta}$
$(1+i)^n=e^{n\delta}$	$v^n=e^{-n\delta}$

Double the force of interest

If you double the force of interest then

$$1+i \to (1+i)^2$$

$$v \to v^2$$

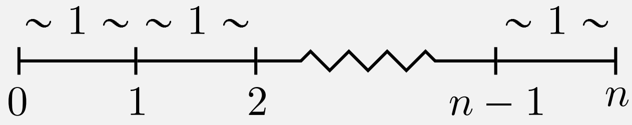
$$i \to 2i+i^2$$

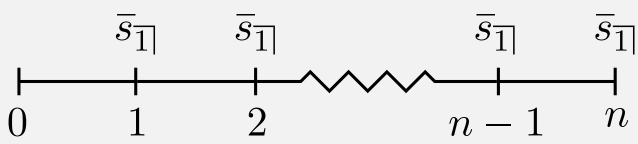
$$d \to 2d-d^2$$

$$\frac{i}{\delta} \to \frac{2i+i^2}{2\delta}$$

Continuous Annuities

$$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta} = \frac{i}{\delta} \cdot \frac{1 - v^n}{i} = \frac{i}{\delta}\, a_{\overline{n}|}$$





$$\bar{s}_{\overline{1}|} = \frac{(1+i)^1 - 1}{\delta}$$

$$= \frac{i}{\delta}$$

Calculator Examples

$$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta} = \frac{i}{\delta} a_{\overline{n}|}$$

$$\bar{a}_{\overline{20}|\, i=0.10} = 8.93248$$

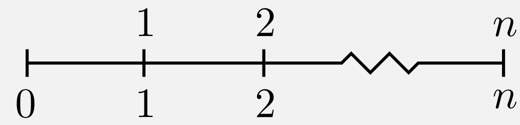
N = 20
I/Y = 10
PMT = 1
FV = 0
CPT PV = −8.51356372
[+|−] × 0.10 ÷ ln(1.10) = 8.932481019

$$\bar{a}_{\overline{20}|\, \delta=0.10} = 8.64665$$

−20 × 0.10 [2nd]e^x = 0.135335283
[+|−] + 1 = 0.864664717
÷ 0.1 = 8.646647168

Increasing Annuities

$(Ia)_{\overline{n|}}$



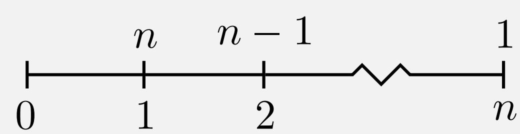
$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$$

$$(Ia)_{\overline{20|}\,i=.10} = 63.92$$

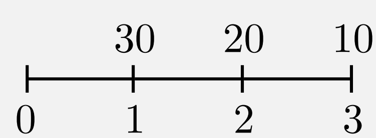
[2nd] BGN
N = 20
I/Y = 10
PMT = 1
FV = −20
CPT PV = −6.392047531
[+|−] ÷ 0.1 = 63.92047531

Decreasing Annuities

$(Da)_{\overline{n|}}$



$10(Da)_{\overline{3|}}$



$$(Da)_{\overline{n|}} = \frac{n - a_{\overline{n|}}}{i}$$

$$(Da)_{\overline{20|}\,i=.10} = 114.864$$

N = 20
I/Y = 10
PMT = 1
FV = 0
CPT PV = −8.51356372
+ 20 = 11.48643628
÷ 0.1 = 114.8643628

Integrals

$$\int_0^\infty v^t dt = \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$$

$$\int_0^\infty t v^t dt = (\bar{I}\bar{a})_{\overline{\infty}|} = \frac{1}{\delta^2}$$

$$\int_0^n t v^t dt = (\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta}$$

$$\int_0^n (n-t) v^t dt = (\bar{D}\bar{a})_{\overline{n}|} = \frac{n - \bar{a}_{\overline{n}|}}{\delta}$$

Integral Example

Example

$$\begin{aligned}
\int_0^5 t e^{-t}\, dt &= (\bar{I}\bar{a})_{\overline{5}|\,\delta=1} \\
&= \frac{\bar{a}_{\overline{5}|\,\delta=1} - 5v^5}{\delta} \\
&= \frac{\frac{1-v^5}{\delta} - 5v^5}{\delta} \\
&= \frac{1 - v^5 - 5v^5}{1} \\
&= 1 - 6v^5 \\
&= 1 - 6e^{-5}
\end{aligned}$$

Exercise

A company is introducing a new product that they think will have a 10-year life cycle, with sales increasing steadily for 5 years, after which sales will decline steadily.

The company feels that the product will be so successful that they will make sales every day of the year. As a result, they model future sales by assuming net cash flows are received continuously over the 10-year horizon at the following rates:

$$\begin{array}{ll} 100t & 0 \leqslant t \leqslant 5 \\ 100(10-t) & 5 \leqslant t \leqslant 10 \end{array}$$

The company requires an annual effective rate on any investment of 12.75%. What is the maximum amount of money the company should spend today to invest in this new product?

Exercise Solution

$$\int_0^5 100t\, e^{-0.12t}\, dt + \int_5^{10} 100(10-t)\, e^{-0.12t}\, dt$$

Let $s = t - 5$, thus $t = s + 5$ and $dt = ds$

$$\begin{aligned} &= 100(\bar{I}\bar{a})_{\overline{5|}} + \int_0^5 100(10-(s+5))e^{-0.12(s+5)}\, ds \\ &= 100(\bar{I}\bar{a})_{\overline{5|}} + e^{-0.12(5)} \int_0^5 100(5-s)e^{-0.12s}\, ds \\ &= 100(\bar{I}\bar{a})_{\overline{5|}} + v^5 \cdot 100(\bar{D}\bar{a})_{\overline{5|}} \\ &= 100\left(\frac{\bar{a}_{\overline{5|}} - 5v^5}{\delta}\right) + 100\left(\frac{5 - \bar{a}_{\overline{5|}}}{\delta}\right) \cdot v^5 \\ &\approx 1414 \end{aligned}$$

$$\delta = \ln(1.1275) = 0.12 \qquad v^5 = \frac{1}{1.1275^5} = 0.5488 \qquad \bar{a}_{\overline{5|}} = \frac{1-v^5}{\delta} = 3.76$$