B.1 Solutions

Last updated October 8, 2013

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1. Since $\omega = 105$

$$ {}_{10|20}q_{25} = \frac{20}{105-25} = \boxed{\frac{1}{4}} $$

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2. When you double the force of mortality, the probability of survival is squared

$$ p'_x = p_x^2 = (1-q_x)^2 $$

The adjusted mortality rate is

$$ \begin{aligned} q'_x &= 1 - p'_x \\ &= 1 - (1-q_x)^2 \\ &= 1 - (1 - 2q_x + q_x^2) \\ &= \boxed{2q_x - q_x^2} \end{aligned} $$

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4. We have DML with $\omega = 100$

$$ {}_{10}p_{50} = \frac{40}{50} = \boxed{0.8} $$

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5. First write ${}_tp_x$ in terms of the expected number of survivors ${}_tp_x = \frac{\ell_{x+t}}{\ell_x}$

For A we have

$$ \begin{aligned} {}_{t|u}q_x - {}_{t+u}p_x &= \frac{\ell_{x+t} - \ell_{x+t+u}}{\ell_x} - \frac{\ell_{x+t+u}}{\ell_x} \\ &= \frac{\ell_{x+t} - 2\ell_{x+t+u}}{\ell_x} \end{aligned} $$

For B we have

$$ {}_{t|u}q_x - {}_tq_x + {}_{t+u}p_x = \frac{\ell_x - \ell_{x+t+u}}{\ell_x} - \frac{\ell_x - \ell_{x+t}}{\ell_x} + \frac{\ell_{x+t+u}}{\ell_x} $$

$$= \frac{\ell_{x+t}}{\ell_x}$$
$$= {}_tp_x$$

We can stop here since $\boxed{\text{B}}$ is the answer. C and D don't even make sense because the dominator is not the same for each term.

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6. Key fact

$$\overset{\circ}{e}_{39} = \frac{T_{39}}{\ell_{39}}$$

We can use ℓ_{38} to find ℓ_{39}

$$\ell_{39} = \ell_{38} - d_{38}$$
$$m_{38} = 0.021 = \frac{d_{38}}{L_{38}} \Rightarrow d_{38} = 0.021(2475) = 51.975$$
$$\ell_{39} = \ell_{38} - 51.975 = 2448.025$$

We can use T_{38} to find T_{39}

$$T_{39} = T_{38} - L_{38} = 95000 - 2475 = 92525$$

Finally

$$\overset{\circ}{e}_{39} = \frac{92525}{2448.025} = \boxed{37.80}$$

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7. 1 is clearly true. For 2, we need the numerator to be the number of deaths b/w ages $x + t$ and $x + t + u = \ell_{x+t} - \ell_{x+t+u}$, but the numerator given is the negative of that, so 2 if false. 3 is also clearly true.

8.

$$p = \frac{\ell_{70} - \ell_{80}}{\ell_{60}} = \boxed{0.33}$$

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9. We are given DML with $\omega = 100$

$$e_{90} = \frac{100 - 90}{2} - \frac{1}{2} = \boxed{4.5}$$

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10. Key facts

$$\overset{\circ}{e}_x = \frac{\omega - x}{2}$$

$$\text{Var}[T(x)] = \frac{(\omega - x)^2}{12}$$

So for $x = 20$

$$\begin{aligned} \mathring{e}_{20} &= \frac{\omega - 20}{2} = 45 \\ \omega &= 110 \end{aligned}$$

$$\text{Var}[T(20)] = \frac{(110 - 20)^2}{12} = \boxed{675}$$

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11. 1 is false, $\mathring{e}_x = e_x + \frac{1}{2}$. 2 is clearly true. 3 is also true, since

$$\int_0^\infty \ell_{x+t}\, dt = T_x$$

and

$$\mathring{e}_x = \frac{T_x}{\ell_x}$$

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12. $\bar{a}_{\overline{20}|} = 1.4 \cdot \bar{a}_{\overline{10}|}$ gives us

$$\begin{aligned} \frac{1 - v^{20}}{\delta} &= 1.4 \cdot \frac{1 - v^{10}}{\delta} \\ v^{20} - 1.4v^{10} + 0.4 &= 0 \\ v^{10} &= 0.4 \\ e^{-10\delta} &= 0.4 \end{aligned}$$

We need to find ${}_{20|10}q_{20}$

$$\begin{aligned} {}_{20|10}q_{20} &= {}_{20}p_{20} - {}_{30}p_{20} \\ &= e^{-20\delta} - e^{-30\delta} \\ &= 0.4^2 - 0.4^3 \\ &= \boxed{0.096} \end{aligned}$$

13.

$${}_{2|2}q_{15} = {}_{2}p_{15} - {}_{4}p_{15}$$

$$= e^{-2(0.0012)} - e^{-4(0.0012)}$$
$$= \boxed{0.00239}$$

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14. This is a tricky problem. The elegant solution realizes that total number of drivers is just 1000 $\mathring{e}_{15:\overline{50}|}$. In other words, the number of drivers is equal to the number of new drivers times the average time each driver lives.

$$\begin{aligned}
\mathring{e}_{15:\overline{50}|} &= \mathring{e}_{15} - {}_{50}p_{15} \cdot \mathring{e}_{65} \\
&= \frac{100-15}{2} - \frac{100-65}{100-15} \cdot \frac{100-65}{2} \\
&= 35.294
\end{aligned}$$

$$1000(35.294) = \boxed{35294}$$

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15. First find the adjust probability of survival in terms of c

$$p_{30} = \exp\left(-\int_0^1 \mu(30+t)\,dt\right)$$

$$\begin{aligned}
p'_{30} &= \exp\left(-\int_0^1 (\mu(30+t) - c)\,dt\right) \\
&= \exp\left(-\int_0^1 \mu(30+t)dt\right)\exp\left(\int_0^1 c\,dt\right) \\
&= p_{30} \cdot e^c \\
&= 0.95 \cdot e^c
\end{aligned}$$

We want to reduce q_{30} by 25%

$$\begin{aligned}
q_{30} &= 1 - 0.95 = 0.05 \\
q'_{30} &= 0.05 \times 0.75 = 1 - p'_{30} \\
0.0375 &= 1 - 0.95e^c \\
c &= \boxed{0.013}
\end{aligned}$$

16.

$$\mu(x) = -\frac{s'(x)}{s(x)}$$
$$= -\frac{\exp\left(-\frac{x^3}{12}\right) \cdot \left(-\frac{3x^2}{12}\right)}{\exp\left(-\frac{x^3}{12}\right)}$$
$$= \boxed{\frac{x^2}{4}}$$

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17. We have DML with $\omega = 125$

$$\mathring{e}_{50} = \frac{T_{50}}{\ell_{50}}$$
$$\frac{125 - 50}{2} = \frac{T_{50}}{1000 \cdot \frac{125-50}{125}}$$
$$T_{50} = \boxed{22500}$$

18.

$$p_{80} = 0.7$$
$$p'_{80} = (0.7)^{\frac{1}{2}} = 0.83666$$
$$q'_{80} = 1 - 0.83666 = \boxed{0.16334}$$

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19. $s(x) = 1 - 0.01x = \frac{100-x}{100} \Rightarrow$ DML with $\omega = 100$. Under DML median future lifetime = mean future lifetime.

$$\mathring{e}_{10} = \frac{100 - 10}{2} = \boxed{45}$$

20.

$${}_2q_{21} = \frac{\ell_{21} - \ell_{23}}{\ell_{21}}$$

$$\begin{aligned} \ell_{21} &= \ell_{20} - d_{20} = 30000 - 1200 = 28800 \\ \ell_{23} \times p_{23} &= \ell_{24} \\ \ell_{23} &= \frac{23900}{0.93} = 25699 \\ {}_2q_{21} &= \frac{28800 - 25699}{28800} = \boxed{0.1077} \end{aligned}$$

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21. We have DML with $\omega = 100$

$$\mathrm{Var}[T(20)] = \frac{(\omega - x)^2}{12} = \frac{(100-20)^2}{12} = \boxed{533.33}$$

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23. Survival follows DML with $\omega = 72$. First we need to find the expected future lifetime of a hen aged 12 months

$$\mathring{e}_{12} = \frac{72 - 12}{2} = 30$$

The total number of eggs is the number of hens times the number of eggs per month times the expected future lifetime

$$100 \times 30 \times 30 = \boxed{90{,}000}$$

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24. You actually don't need to know anything about Weibull's law. They give you all the information you need.

$$\begin{aligned} \frac{d}{dt}\,{}_tq_x &= {}_tp_x\,\mu(x+t) \\ &= \frac{s(x+t)}{s(x)} \cdot \mu(x+t) \\ &= \frac{\exp(-0.015(35)^{1.1})}{\exp(-0.015(25)^{1.1})} \cdot 0.0165(35)^{0.1} \\ &= \boxed{0.0186743} \end{aligned}$$

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25. We have DML with $\omega = 70$. The expected density of deaths is

$$\ell_x\,\mu(x) = \ell_0\,s(x)\,\mu(x)$$

$$= 1000 \cdot \frac{70-30}{70} \cdot \frac{1}{70-30} = \boxed{14.286}$$

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26. $s(x) = e^{-0.05x}$ implies CF with $\mu = 0.05$. Under CF $m_x = \mu = \boxed{0.05}$.

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27. We have DML with $\omega = 100$

$$\begin{aligned}
\mathring{e}_{25} &= \frac{T_{25}}{\ell_{25}} \\
\frac{100-25}{2} &= \frac{T_{25}}{500 \cdot \frac{100-25}{100}} \\
T_{25} &= \boxed{14062.5}
\end{aligned}$$

28.

$$\begin{aligned}
p_{75} &= 0.88 \\
p'_{75} &= (0.88)^2 = 0.7744 \\
q'_{75} &= 1 - 0.7744 = \boxed{0.2256}
\end{aligned}$$

29.

$$10000\,{}_{20|10}q_{30}^{\omega=100} + 10000\,{}_{10|10}q_{40}^{\omega=90} = 10000 \cdot \frac{10}{70} + 10000 \cdot \frac{10}{50} = \boxed{3429}$$

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30. First find ℓ_x

$$\begin{aligned}
\mu(x) &= -1 \cdot \frac{-1}{10+x} \\
\therefore \ell_x &= (10+x)^{-1} \\
{}_{5|15}q_0 &= \frac{\frac{1}{15} - \frac{1}{30}}{\frac{1}{10}} = \boxed{\frac{1}{3}}
\end{aligned}$$

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31. Under CF we have

$$\mathring{e}_{80} = \frac{1}{\mu} = 20$$

$$T_{80} = \ell_{80} \overset{\circ}{e}_{80} = 10000e^{-80(.05)}(20) = 3663$$

Under DML we have

$$\begin{aligned} \overset{\circ}{e}_{80} &= \frac{100-80}{2} = 10 \\ T_{80} &= \ell_{80} \overset{\circ}{e}_{80} = 1000\left(\frac{20}{100}\right)(10) = 20000 \end{aligned}$$

The difference is

$$20000 - 3663 = \boxed{16337}$$

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33. Let's find $s(x)$ first

$$\begin{aligned} s(x) = {}_{x}p_0 &= \exp\left(-\int_0^x \mu(y)dy\right) \\ &= \exp\left(-\int_0^x 2y\,dy\right) \\ &= \left(-\,y^2\right]_0^x\Big) \\ &= \exp\left(-x^2\right) \end{aligned}$$

That eliminates B, C and D. Now we find $F(x)$.

$$F(x) = 1 - s(x) = 1 - e^{-x^2}$$

Therefore $\boxed{\text{E}}$ is the correct answer.

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34. We want to find ${}_{5|}q_{30}$

$$\begin{aligned} {}_{5|}q_{30} &= {}_{5}p_{30} - {}_{6}p_{30} \\ &= \exp\left(-\int_{30}^{35} 0.02\,x^{0.5}\,dx\right) - \exp\left(-\int_{30}^{36} 0.02\,x^{0.5}\,dx\right) \\ &= \exp\left(-\frac{0.02}{1.5}x^{1.5}\Big]_{30}^{35}\right) - \exp\left(-\frac{0.02}{1.5}x^{1.5}\Big]_{30}^{36}\right) \\ &= 0.5656 - 0.5020 \end{aligned}$$

$$= \boxed{0.0636}$$

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35. The ratio is

$$\frac{\overset{\circ}{e}_{x:\overline{10|}}}{\overset{\circ}{e}_{y:\overline{10|}}} = \frac{\overset{\circ}{e}_x \left(1 - {}_{10}p_x\right)}{\overset{\circ}{e}_y \left(1 - {}_{10}p_y\right)} = \frac{\frac{1}{0.1}\left(1 - e^{-10(0.1)}\right)}{\frac{1}{0.2}\left(1 - e^{-10(0.2)}\right)} = \boxed{1.462}$$

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36. Simple recursion

$$e_{50} = p_{50}\left(1 + e_{51}\right)$$

$$20 = 0.97\left(1 + e_{51}\right)$$

$$e_{51} = \boxed{19.619}$$

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37. We add up the force of mortality between ages 40 and 42, take the negative and take the exponential

$${}_2p_{40} = \exp\left(-\int_{40}^{42} \tfrac{1}{100}\, t\, dt\right) = \boxed{0.44}$$

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39. First simplify the expression

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6 = {}_5p_1 \cdot q_6 = {}_{5|1}q_1 = \frac{\ell_6 - \ell_7}{\ell_1}$$

Now find ℓ_1, ℓ_6 and ℓ_7

$$\ell_1 = 1000(0.875) = 875$$
$$\ell_6 = 200 - 120 = 80$$
$$\ell_7 = 20$$

Plugging these back into the first equation we have

$$p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot p_5 \cdot q_6 = \frac{80 - 20}{875} = \boxed{0.06857}$$

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40. We are given MDML with $\omega = 100$ and $a = \frac{1}{2}$

$${}_{40}p_{25} = \left(\frac{35}{75}\right)^{1/2} = \boxed{0.683}$$

41. The force of mortality is the derivative of the expected number of survivors divided by the expected number of survivors

$$\mu(x) = \frac{-\frac{d}{dx}\,\ell_x}{\ell_x}$$

$$\mu(x) = \frac{100\left(\frac{2}{3}\right)(k-0.5x)^{-\frac{1}{3}}(0.5)}{100\,(k-0.5x)^{\frac{2}{3}}} = \tfrac{1}{3}\,(k-0.5x)^{-1}$$

For $x = 50$

$$\begin{aligned}\tfrac{1}{48} &= \tfrac{1}{3}\,(k-25)^{-1}\\ k &= \boxed{41}\end{aligned}$$

42. The sum of the force of mortality over all possible ages must equal infinity and the sum of the pdf over all possible ages must equal 1.

(i) $\int_0^\infty (1+x)^{-3}dx = \frac{1}{2} \neq \infty \;\therefore$ invalid.

(ii) $\int_0^\infty 0.05(1.01)^x dx = 0.05(1.01)^x/\ln(1.01)\big]_0^\infty = \infty, \therefore$ valid. Note this is Makeham's.

(iii) $\int_0^\infty e^{-\frac{x}{2}}dx = -2e^{-\frac{x}{2}}\Big]_0^\infty = 2 \neq 1 \;\therefore$ invalid. This is almost exponential, but needed a coefficient of $\frac{1}{2}$.

46. If we let $\ell_x = 10000 - x^2$ then $-\frac{d}{dx}\ell_x = 2x$ and $\mu(x) = \frac{2x}{10000-x^2}$ as given. Thus

$$q_x = \frac{\ell_x - \ell_{x+1}}{\ell_x} = \frac{10000 - x^2 - \left(10000 - (x+1)^2\right)}{10000 - x^2} = \boxed{\frac{2x+1}{10000-x^2}}$$

48. Build a table

x	ℓ_x
60	100,000
61	98,000
62	95,844

Thus $\ell_{61.5} = \frac{1}{2}(98000 + 95844) = 96922$. We want the average lifetime of a 60 year old limited to 1.5 years so let's consider the three distinct cases (1) die between ages 60 and 61, (2) die between ages 61 and 61.5 and (3) live to age 61.5. The average lifetime limited to 1.5 years under those three cases are (1) 0.5, (2) $1 + 0.5/2 = 1.25$ and (2) 1.5.

So the average lifetime for one of these 60 years olds is the total lifetime limited to 1.5 years divided by the number of 60 year olds

$$\frac{0.5(100{,}000 - 98{,}000) + 1.25(98{,}000 - 96{,}922) + 1.5(96{,}922)}{100{,}000} = \boxed{1.4773}$$

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49. Because $\delta = 0$, $\bar{a}_{\overline{T|}} = T$. Thus

$$\begin{aligned}\mathrm{Var}[\bar{a}_{\overline{T|}}] &= \mathrm{Var}[T] \\ h &= \mathrm{E}[T^2] - (\mathrm{E}[T])^2 \\ h &= 2g - \left(\mathring{e}_x\right)^2 \\ \mathring{e}_x &= \boxed{\sqrt{2g - h}}\end{aligned}$$

50.We are given $m_{70} = 0.1$

$$0.1 = \frac{d_{70}}{L_{70}} = \frac{\ell_{70} - \ell_{71}}{L_{70}}$$

We are also given $a(70) = 0.45$

$$\begin{aligned}0.45 &= \frac{L_{70} - \ell_{71}}{\ell_{70} - \ell_{71}} \\ 0.45 &= \frac{L_{70}}{\ell_{70} - \ell_{71}} - \frac{\ell_{71}}{\ell_{70} - \ell_{71}} \\ 0.45 &= \frac{1}{0.1} - \frac{\ell_{71}}{1055 - \ell_{71}} \\ \ell_{71} &= \boxed{955}\end{aligned}$$

52. $d_x = k$ means there are an equal number of deaths each year. In other words, we have DML.

$$\begin{aligned}\mathring{e}_{20:\overline{20}|} &= {}_{20}p_{20}(20) + {}_{20}q_{20}(10) \\ 18 &= \frac{\omega - 40}{\omega - 20}(20) + \frac{20}{\omega - 20}(10) \\ \omega &= 120\end{aligned}$$

Now we can find the probability

$$_{30|10}q_{30} = \frac{10}{120 - 30} = \boxed{0.111}$$

53. We are given $\mathring{e}_{36:\overline{28}|}$ so let's try to write that in terms of the integral we desire

$$\begin{aligned}\mathring{e}_{36:\overline{28}|} &= \int_0^{28} t\, {}_tp_{36}\, \mu(36+t)\, dt + 28\, {}_{28}p_{36} \\ 24.67 &= \int_0^{28} t\, {}_tp_{36}\, \mu(36+t)\, dt + 28\left[\frac{(100-64)^{0.5}}{(100-36)^{0.5}}\right] \\ \boxed{3.67} &= \int_0^{28} t\, {}_tp_{36}\, \mu(36+t)\, dt\end{aligned}$$

55. in general $\frac{d}{dt}\, {}_tq_x = {}_tp_x\, \mu(x+t)$. And under UDD ${}_tq_x = t\, q_x$ thus

$$_tp_x\, \mu(x+t) = \tfrac{d}{dt}\, t\, q_x = q_x$$

Thus

$$q_x = \boxed{\tfrac{12}{49}}$$

56. All of these functions are non-negative so we just need to check that $s(\infty) = 0$

(i) $s(x) = e^{-\int_0^x Bc^t\, dt} = e^{-\frac{Bc^t}{\ln c}\Big]_0^x} = e^{\frac{B}{\ln c} - \frac{Bc^x}{\ln c}}$

$s(\infty) = e^{\frac{B}{\ln c}} \neq 0$

(ii) $s(x) = e^{-\int_0^x B(t+1)^{-0.5}\, dt} = e^{-2B(t+1)^{0.5}\Big]_0^x} = e^{2B - 2B(x+1)^{0.5}}$

$s(\infty) = e^{-\infty} = 0$ ✓

(iii) $s(x) = e^{-\int_0^x k(t+1)^n\,dt} = e^{-\frac{k(t+1)^{n+1}}{n+1}\Big]_0^x} = e^{\frac{k-k(x+1)^{n+1}}{n+1}}$

$s(\infty) = e^{-\infty} = 0$ ✓

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57. Under DML $\mu(x) = \dfrac{1}{\omega - x}$

(i) $\dfrac{q_{x-1}}{p_{x-1}} = \dfrac{\frac{1}{\omega-(x-1)}}{\frac{\omega-(x-1)-1}{\omega-(x-1)}} = \dfrac{1}{\omega - x}$ ✓

(ii) $\dfrac{1}{2\,m(x)} = \dfrac{1}{2\left(\frac{\omega-x}{2}\right)} = \dfrac{1}{\omega - x}$ ✓

(iii) $\dfrac{m_x}{1+0.5\,m_x} = \dfrac{\frac{d_x}{L_x}}{1+0.5\frac{d_x}{L_x}} = \dfrac{d_x}{L_x + 0.5d_x} = \dfrac{d_x}{\ell_x - 0.5d_x + 0.5d_x} = q_x = \dfrac{1}{\omega - x}$ ✓

59.

(i) ${}_{1|2}q_{36} = \dfrac{\ell_{37} - \ell_{39}}{\ell_{36}} = \dfrac{96-87}{99} = 0.091$ ✓

(ii) $m_{37} = \dfrac{d_{37}}{L_{37}} = \dfrac{d_{37}}{\ell_{38} + 0.5d_{37}} = \frac{4}{92+0.5(4)} = 0.043$ ✓

(iii) $\ell_{38.5} = \frac{1}{2}(92) + \frac{1}{2}(87) = 89.5$

$\ell_{38+5/6} = \frac{1}{6}(92) + \frac{5}{6}(87) = 87.833$

${}_{1/3}q_{38.5} = \dfrac{\ell_{38.5} - \ell_{38+5/6}}{\ell_{38.5}} = \dfrac{89.5 - 87.833}{89.5} = 0.0186 \neq 0.021$

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60. First find q_{45} using $\mu(45.5)$ and the assumption of UDD

$$\mu(45.5) = \frac{q_{45}}{1 - 0.5q_{45}} = 0.5$$

$$q_{45} = \frac{1}{2.5}$$

Now calculate $\overset{\circ}{e}_{45:\overline{1}|}$

$$\overset{\circ}{e}_{45:\overline{1}|} = p_{45} + 0.5q_{45}$$

$$= 1 - \frac{1}{2.5} + 0.5\left(\frac{1}{2.5}\right)$$
$$= \boxed{0.8}$$

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61. Build a table

x	ℓ_x
70	100
71	96
72	91.776

Thus

$$\ell_{71.5} = \frac{96 + 91.776}{2} = 93.888$$

Now we can break up the 100 people age 70 into three distinct cases (1) those who die between ages 70 and 71, (2) those who die between ages 71 and 71.5 and (3) those that live to age 71.5. The average lifetime for those three cases limited to 1.5 years is (1) 0.5, (2) 1 + 0.5/2 = 1.25 and (3) 1.5. Therefore

$$\mathring{e}_{70:\overline{1.5}|} = \frac{0.5(4) + 1.25\,(96 - 93.888) + 1.5(93.888)}{100} = \boxed{1.45472}$$

62.

$$\begin{aligned}
\mu^*(x+s) &\geq \mu(x+s) \\
\frac{q_x}{1 - sq_x} &\geq -\ln p_x = -\ln(1 - .25) = -\ln\tfrac{3}{4} = \ln\tfrac{4}{3} = 0.2877 \\
\frac{0.25}{1 - 0.25s} &\geq 0.2877 \\
s &\geq \boxed{0.5242}
\end{aligned}$$

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63. Add up the force of mortality between ages 0 and 0.5, take the negative and take the exponential

$$_{0.5}p_0 = \exp\left(-\int_0^{0.5} A + e^t\,dt\right)$$

$$0.5 = \exp\left(-\left[At + e^t\right]_0^{0.5}\right)$$

$$0.5 = \exp\left(-0.5A - e^{0.5} + 1\right)$$

$$-\ln 0.5 = 0.5A + e^{0.5} - 1$$

$$A = \boxed{0.09}$$

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64. Under DML $\mu(x) = \frac{1}{\omega - x}$

(i) $\dfrac{1}{2\ \overset{\circ}{e}_x} = \dfrac{1}{2\left(\frac{\omega-x}{2}\right)} = \dfrac{1}{\omega - x}$ ✓

(ii) ${}_{n|}q_x = \dfrac{1}{\omega - x}$ ✓

(iii) $\dfrac{m_x}{1 + 0.5\,m_x} = \dfrac{\frac{d_x}{L_x}}{1 + 0.5\frac{d_x}{L_x}} = \dfrac{d_x}{L_x + 0.5d_x} = \dfrac{d_x}{\ell_x - 0.5d_x + 0.5d_x} = q_x = \dfrac{1}{\omega - x}$ ✓

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65. A valid survival function must satisfy three conditions

(i) $s(0) = 1$

(ii) $s(\infty) = 0$

(iii) non-increasing

You can easily validate the first two conditions for all three functions. It is also easily shown that functions 2 and 3 are non-increasing. For function 1 if we calculate $s(0.5)$ we get $e^{0.21} = 1.2337$ and since $s(0) = 1$, function 1 fails the non-increasing condition.

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68. We have MDML with $a = r$

$$\mu(y) = 0.1$$

$$\frac{r}{\omega - y} = 0.1$$

$$10r = \omega - y$$

We are also given $\mathring{e}_y = 8.75$

$$\begin{aligned}
\mathring{e}_y &= 8.75 \\
\frac{\omega - y}{r+1} &= 8.75 \\
\omega - y &= 8.75(r+1)
\end{aligned}$$

Putting the two equations together we have

$$\begin{aligned}
10r &= 8.75(r+1) \\
r &= \boxed{7}
\end{aligned}$$

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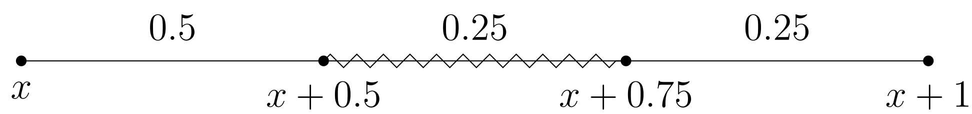
70. First we need to find q_x. We are given ${}_{0.75}p_x = 0.25$ thus ${}_{0.75}q_x = 1 - 0.25 = 0.75$. Since the assumption is UDD, then through 75% of the year 75% of the deaths have occurred:

$$\begin{aligned}
{}_{0.75}q_x &= 0.75 \cdot q_x \\
0.75 &= 0.75 \cdot q_x \\
q_x &= 1
\end{aligned}$$

Under UDD we use the length of the squiggly line times q_x divide by 1 minus the length of the line to the left of the squiggly line times q_x

(i) Draw a picture



$${}_{0.25}q_{x+0.5} = \frac{0.25q_x}{1 - 0.5q_x} = \frac{0.25(0.75)}{1 - 0.5(0.75)} = 0.5 \checkmark$$

(ii) For this one we can draw a picture or just think that for half the year half the deaths have occured

$${}_{0.5}q_x = 0.5\, q_x = 0.5(0.75) = 0.5 \checkmark$$

(iii) You can either memorize $\mu(x+t)$ under UDD or you can quickly derive it:

$$\begin{aligned} \tfrac{d}{dt}\,{}_tq_x &= {}_tp_x\,\mu(x+t) \\ q_x &= (1-t\,q_x)\ \mu(x+t) \\ \mu(x+t) &= \frac{q_x}{1-t\,q_x} \\ \mu(x+0.5) &= \frac{1}{1-0.5} = 2 \neq 0.5 \end{aligned}$$

---

71. For this problem you need to have the force of mortality functions memorized for Gompertz and Weibull:

$$\begin{aligned} \mu^G(x) &= Bc^x \\ \mu^W(x) &= kx^n \end{aligned}$$

Now it is just algebra

$$\begin{aligned} \mu^G(50) &= \mu^W(50) \\ 0.05\left(10^{0.04}\right)^{50} &= 0.1(50)^n \\ 50 &= 50^n \\ n &= \boxed{1} \end{aligned}$$

---

72. This is MDML

$$\mu(x)\cdot \mathring{e}_x = \frac{\alpha}{\omega-x}\cdot\frac{\omega-x}{\alpha+1} = \boxed{\frac{\alpha}{\alpha+1}}$$

---

74. For the first half of the year the additional mortality is c, then it uniformly decreases to 0. So we need the equation for a straight line from $(0.5, c)$ to $(1, 0)$. The slope is $m = \frac{c-0}{0.5-1} = -2c$. Using the second point we find the y-intercept, $0 = -2c(1) + b$ thus $b = 2c$. So our extra mortality for the last half of the year is $-2ct + 2c$.

$$p'_{35} = \exp\left(-\int_0^1 \mu'(35+t)dt\right)$$

$$= \exp\left(-\int_0^{0.5} \mu + c\,dt\right) \cdot \exp\left(-\int_{0.5}^{1} \mu - 2ct + 2c\,dt\right)$$

$$= \exp\left(-\int_0^{0.5} \mu\,dt\right) \cdot \exp\left(-\int_0^{0.5} c\,dt\right) \cdot \exp\left(-\int_{0.5}^{1} \mu\,dt\right) \cdot \exp\left(-\int_{0.5}^{1} 2c - 2ct\,dt\right)$$

$$= p_{35}\, e^{-0.5c}\, e^{-c+0.75c} = 0.985e^{-0.75c}$$

$$q_{35} = \boxed{1 - 0.985e^{-0.75c}}$$

---

75. The new type is guaranteed to survive to age 5, then it's future lifetime is uniformly distributed from 0 to 35 beyond age 5.

$$\overset{\circ}{e}{}_0^{*} = 5 + \frac{35}{2} = 22.5$$

$$\overset{\circ}{e}{}_0^{*} - \overset{\circ}{e}_x = 22.5 - 20 = \boxed{2.5}$$

---

76. We are given MDML with $\omega = 100$ and $a = 4$

$$\begin{aligned}\frac{T_{40} - T_{60}}{\ell_{30}} &= \frac{\ell_{40}}{\ell_{30}}\left(\frac{T_{40} - T_{60}}{\ell_{40}}\right) \\ &= {}_{10}p_{30}\ \overset{\circ}{e}_{40:\overline{20|}} \\ &= {}_{10}p_{30}\left(\overset{\circ}{e}_{40} - {}_{20}p_{40}\ \overset{\circ}{e}_{60}\right) \\ &= \left(\frac{60}{70}\right)^4 \left(\frac{60}{5} - \left(\frac{40}{60}\right)^4 \cdot \frac{40}{5}\right) \\ &= \boxed{5.624}\end{aligned}$$

---

77. When you see $(k^3 - x)^{1/3}$ you should be thinking this might be MDML. With one step of algebra we confirm this

$$s(x) = \left(\frac{k^3 - x}{k^3}\right)^{1/3} = \left(\frac{\omega - x}{\omega}\right)^a$$

So we have MDML with $\omega = k^3$ and $a = \frac{1}{3}$

$$\mathring{e}_{40} = 2\ \mathring{e}_{80}$$

$$\frac{k^3 - 40}{\frac{1}{3}+1} = 2\left(\frac{k^3 - 80}{\frac{1}{3}+1}\right)$$

$$k^3 = 120$$

Finally

$$\mathring{e}_{60} = \frac{120-60}{\frac{1}{3}+1} = \boxed{45}$$

---

78. This is MDML with $\omega = 100$ and $a = 1/2$, but we don't need to know that to work the problem. Let's start with $F(75)$

$$F(x) = 1 - s(x) = 1 - \frac{\sqrt{100-x}}{10}$$

$$F(75) = 1 - \frac{5}{10} = 0.5$$

Now we find $f(75)$

$$f(x) = F'(x) = \frac{1}{20}(100-x)^{-1/2}$$

$$f(75) = .01$$

Finally we find $\mu(75)$

$$\mu(x) = \frac{f(x)}{s(x)}$$

$$\mu(75) = \frac{.01}{\frac{5}{10}} = \boxed{0.02}$$

---

79. Build a table

y	ℓ_y	d_y
$x+1$	100 (arbitrary)	$100 - 95 = 5$
$x+2$	$85.5 + 9.5 = 95$	$100 \times .095 = 9.5$
$x+3$	$\frac{17.1}{.2} = 85.5$	$100 \times .171 = 17.1$

$$q_{x+1} + q_{x+2} = \frac{5}{100} + \frac{9.5}{95} = \boxed{0.15}$$

---

80. K is the number of complete future birthdays of a life age 96. Completing the table we have

x	d_x	K
96	50	0
97	57	1
98	42	2
99	31	3
100	0	4

Now this is a 1/P problem

$$\begin{aligned}
\mathrm{E}(K) &= \frac{57}{180}(1) + \frac{42}{180}(2) + \frac{31}{180}(3) = 1.3 \\
\mathrm{E}(K^2) &= \frac{57}{180}(1)^2 + \frac{42}{180}(2)^2 + \frac{31}{180}(3)^2 = 2.8 \\
\mathrm{Var}(K) &= 2.8 - 1.3^2 = \boxed{1.11}
\end{aligned}$$

---

81. Rewrite q_x in terms of the survival function, then use the product rule to take the derivate

$$\begin{aligned}
\frac{d}{dx}\, q_x &= \frac{d}{dx}\left(1 - \frac{s(x+1)}{s(x)}\right) \\
&= \frac{d}{dx}\left(-\frac{s(x+1)}{s(x)}\right) \\
&= \frac{-s'(x+1)}{s(x)} - s(x+1)(-1)\left(s(x)^{-2}\right) s'(x) \\
&= \frac{s(x+1)}{s(x)}\left(-\frac{s'(x+1)}{s(x+1)}\right) - \frac{s(x+1)}{s(x)}\left(-\frac{s'(x)}{s(x)}\right) \\
&= p_x\, \mu(x+1) - p_x\, \mu(x) \\
&= \boxed{-p_x\left(\mu(x) - \mu(x+1)\right)}
\end{aligned}$$

82. The first moment is

$$
\begin{aligned}
\mathrm{E}[T(0)] &= \frac{\int_0^\omega \ell_x \, dx}{\ell_0} \\
&= \frac{\int_0^\omega \omega^3 - x^3 \, dx}{\omega^3} \\
&= \frac{\omega^3 x - \frac{x^4}{4}\Big]_0^\omega}{\omega^3} \\
&= \frac{\frac{3}{4}\omega^4}{\omega^3} \\
&= \tfrac{3}{4}\omega
\end{aligned}
$$

The second moment is

$$
\begin{aligned}
\mathrm{E}[T(0)^2] &= \frac{2\int_0^\omega x\ell_x \, dx}{\ell_0} \\
&= \frac{2\int_0^\omega \omega^3 x - x^4 \, dx}{\ell_0} \\
&= 2\left[\frac{\frac{\omega^3 x^2}{2} - \frac{x^5}{5}\Big]_0^\omega}{\omega^3}\right] \\
&= 0.6\omega^2
\end{aligned}
$$

The variance is the second moment minus the first moment squared

$$
\mathrm{Var}[T(0)] = 0.6\omega^2 - \left(\tfrac{3}{4}\omega\right)^2 = \boxed{\tfrac{3}{80}\omega^2}
$$

83. We add up the force mortality between ages 35 and 45, take the negative and take the exponential

$$
{}_{10}p_{35} = e^{-\int_{35}^{45} \mu(x)\, dx}
$$

$$
0.81 = e^{-\int_{35}^{45} kx\, dx}
$$

$$0.81 = e^{-\frac{kx^2}{2}\Big]_{35}^{45}}$$

$$0.81 = e^{-400k}$$

Now do the same for ${}_{20}p_{40}$

$$\begin{aligned}
{}_{20}p_{40} &= e^{-\int_{40}^{60} kx\,dx} \\
&= e^{-\frac{kx^2}{2}\Big]_{40}^{60}} \\
&= e^{-1000k} \\
&= \left(e^{-400k}\right)^{\frac{1000}{400}} \\
&= (0.81)^{\frac{5}{2}} \\
&= \boxed{0.59}
\end{aligned}$$

---

84. ${}_{t|}q_x = 0.10$ means that each year 10% of the original ℓ_x lives die each year. In other words, the number of deaths is the same each year. The future lifetime of x follows a discrete uniform distribution over 10 years, thus

$${}_2p_{x+5} = \frac{3}{5}$$

If you want to think of this in terms of DML, then it is Discrete DML with $\omega = x + 10$. You might be tempted to say $\omega = 10$, but this life is already age x and his future lifetime is at most 10 years so the limiting age is $x + 10$ not 10.

$${}_2p_{x+5} = \frac{x + 10 - (x + 5) - 2}{x + 10 - (x + 5)} = \boxed{\frac{3}{5}}$$

85.

$$\begin{aligned}
\overset{\circ}{e}_{30} &= \frac{\omega - 30}{2} = 30 \\
\omega &= 90 \\
q_{90} &= \boxed{\frac{1}{60}}
\end{aligned}$$

86.

$$\begin{aligned} q_x^* &= 1 - p_x^* \\ &= 1 - (1 - q_x)^{0.5} \\ &= \boxed{1 - \sqrt{1 - q_x}} \end{aligned}$$

87.

$$\begin{aligned} {}_m p_y &= \frac{s(y+m)}{s(y)} \\ 0.5 &= \frac{\frac{1}{1+y+m}}{\frac{1}{1+y}} \\ 0.5 &= \frac{1+y}{1+y+m} \\ 2(1+y) &= 1+y+m \\ \boxed{1+y} &= m \end{aligned}$$

88.

$$\begin{aligned} \hat{p}_x &= e^{-\int_0^1 \hat{\mu}(x+t)\,dt} \\ 1 - \hat{q}_x &= e^{-\int_0^1 \mu(x+t)-k\,dt} \\ 1 - 0 &= e^{-\int_0^1 \mu(x+t)\,dt} \cdot e^{\int_0^1 k\,dt} \\ 1 &= p_x\, e^k \\ \ln \tfrac{1}{p_x} &= k \\ \ln p_x^{-1} &= k \\ \boxed{-\ln p_x} &= k \end{aligned}$$

89.

(i) ${}_xp_0 = s(x) = 1 - F(x) = \frac{1}{x+1}$ ✓

(ii) $\mu(x) = -\frac{s'(x)}{s(x)} = -\frac{-\left(\frac{1}{x+1}\right)^2}{\frac{1}{x+1}} = \frac{1}{x+1}$

$\mu(49) = 0.02$ ✓

(iii) ${}_{10}p_{39} = \frac{s(49)}{s(39)} = \frac{\frac{1}{50}}{\frac{1}{40}} = \frac{4}{5} = 0.8$ ✓

---

90. Use the variance to find ω

$$\begin{aligned}\mathrm{Var}[T(15)] &= 675 \\ \frac{(\omega - 15)^2}{12} &= 675 \\ \omega &= 105\end{aligned}$$

Now find the expected future lifetime

$$\mathring{e}_{25} = \frac{105 - 25}{2} = \boxed{40}$$

---

91. Add up the force between ages y and $y+1$, take the negative and take exponential

$$p_y = \exp\left(-\int_0^1 \mu(y+t)\,dt\right)$$

But notice that $\mu(y+t) = 2\mu(x+t)$ so

$$\begin{aligned}&= \exp\left(-\int_0^1 2\mu(x+t)\,dt\right) \\ &= \left[\exp\left(-\int_0^1 \mu(x+t)\,dt\right)\right]^2 \\ &= [1 - 0.04]^2 \\ &= 0.9261\end{aligned}$$

$$q_y = 1 - 0.9261 = \boxed{0.0784}$$

93.

$$
\begin{aligned}
S &= \tfrac{2}{3}R \\
1 - e^{-\int_0^1 \mu_{x+t}\,dt}\, e^{-\int_0^1 k\,dt} &= \tfrac{2}{3}\left(1 - p_x\right) \\
1 - p_x\, e^k &= \tfrac{2}{3} q_x \\
1 - \tfrac{2}{3} q_x &= p_x\, e^k
\end{aligned}
$$

$$\boxed{\ln\left(\frac{1 - \frac{2}{3}q_x}{p_x}\right)} = k$$

---

95. Note that is MDML with $\omega = k^2$ and $a = \frac{1}{2}$. This can be seen with a little algebra

$$s(x) = \frac{\sqrt{k^2 - x}}{k} = \left[\frac{k^2 - x}{k^2}\right]^{\frac{1}{2}} = \left[\frac{\omega - x}{\omega}\right]^a$$

Under MDML $\mathring{e}_x = \frac{\omega - x}{a+1}$

$$
\begin{aligned}
\mathring{e}_{40} &= 2\ \mathring{e}_{80} \\
\frac{k^2 - 40}{1.5} &= 2\left(\frac{k^2 - 80}{1.5}\right) \\
120 &= k^2
\end{aligned}
$$

Now find $\mathring{e}_{60}$

$$\mathring{e}_{60} = \frac{120 - 60}{1.5} = \boxed{40}$$

---

96. Let m be the median future lifetime of (20), then

$$
\begin{aligned}
0.5 &= {}_m p_{20} \\
0.5 &= \exp\left(-\int_{20}^{20+m} \sqrt{\frac{1}{80 - x}}\, dx\right) \\
0.5 &= \exp\left(-\int_{20}^{20+m} (80 - x)^{-\frac{1}{2}}\, dx\right)
\end{aligned}
$$

$$0.5 = \exp\left(2\,(80-x)^{\frac{1}{2}}\Big]_{20}^{20+m}\right)$$

$$0.5 = \exp\left(2\left[\sqrt{60-m}-\sqrt{60}\right]\right)$$

$$m = \boxed{5.25}$$

---

97. We are given ${}_tp_x = (0.8)^2$ which implies CF with $p = 0.8$.

$$p = e^{-\mu}$$
$$0.8 = e^{-\mu}$$
$$\mu = -\ln 0.8$$

To find T_{x+1} we will use

$$\overset{\circ}{e}_{x+1} = \frac{T_{x+1}}{\ell_{x+1}}$$

So we need ℓ_{x+1}

$$\ell_{x+2} = \ell_{x+1}\, p_{x+1}$$
$$6.4 = \ell_{x+1} 0(.8)$$
$$\ell_{x+1} = 8$$

So finally

$$T_{x+1} = \overset{\circ}{e}_{x+1} \cdot \ell_{x+1} = \frac{1}{-\ln 0.8} \cdot 8 = \boxed{35.9}$$

---

98. Under CF the future lifetime is distributed exponential with mean $1/\mu$. Since the pdf follows the exponential distribution we have CF with $\mu = 2$.

$$\overset{\circ}{e}_x = \frac{1}{\mu} = \boxed{\frac{1}{2}}$$

---

99. Under CF the future lifetime is distributed exponential with mean $1/\mu$. Since the pdf follows the exponential distribution we have CF with $\mu = 2$.

$$\mathrm{Var}[T] = \frac{1}{\mu^2} = \boxed{\frac{1}{4}}$$

100. Under CF the future lifetime is distributed exponential with mean $1/\mu$. Since the pdf follows the exponential distribution we have CF with $\mu = 2$. Let m be the median, then

$$\begin{aligned} {}_m p_x &= 0.5 \\ e^{-m\mu} &= 0.5 \\ e^{-2m} &= 0.5 \\ m &= \frac{-\ln 0.5}{2} \\ m &= \boxed{\frac{\ln 2}{2}} \end{aligned}$$

---

101. Under CF the future lifetime is distributed exponential with mean $1/\mu$. Since the pdf follows the exponential distribution we have CF with $\mu = 2$. Under CF $m_x = \mu = 2$. If you didn't memorize that, then you can derive it quickly

$$m_x = \frac{d_x}{L_x} = \frac{\int_0^1 \ell_{x+t}\, \mu(x+t)\, dt}{\int_0^1 \ell_{x+t}\, dt} = \frac{\mu \int_0^1 \ell_{x+t}\, dt}{\int_0^1 \ell_{x+t}\, dt} = \boxed{\mu}$$

---

104. First note that ${}_t p_x\, \mu_{x+t} = \frac{d}{dt}\, {}_t q_x$ and under UDD ${}_t q_x = t\, q_x$ so ${}_t p_x\, \mu_{x+t} = q_x$. This is not surprising since ${}_t p_x\, \mu_{x+t}$ is the pdf and we expect this to be uniform (i.e. not a function of t) over 0 to 1.

$${}_{3/4} p_x + \tfrac{1}{2} \left({}_{1/2} p_x\, \mu_{x+1/2} \right) = 1 - \tfrac{3}{4} q_x + \tfrac{1}{2} q_x = 1 - \tfrac{1}{4} q_x = \boxed{{}_{1/4} p_x}$$

105.

$$q_{32} = 1 - p_{32} = 1 - \frac{s(33)}{s(32)} = \boxed{0.00724}$$

---

106. For the actual force of mortality we have

$$\mu_x = \frac{-\frac{d}{dx} \ell_x}{\ell_x} = \frac{2000 \left(\frac{x}{100}\right) \left(\frac{1}{100}\right)}{1000 \left[1 - \left(\frac{x}{100}\right)\right]}$$

$$\mu_{52.25} = 0.013445$$

For the approximation we assume UDD

$$\mu_x^L = \frac{-\frac{d}{dt}\,{}_tp_x}{{}_tp_x} = \frac{\frac{d}{dt}\,{}_tq_x}{{}_tp_x} = \frac{q_x}{1 - t\,q_x}$$

Find q_{50}

$$q_{50} = \frac{\ell_{50} - \ell_{51}}{\ell_{50}} = .013467$$

Finally

$$\mu_{50.25} - \mu_{50.25}^L = 0.013445 - \frac{0.013467}{1 - 0.25(0.13467)} = \boxed{-0.000067}$$

---

107. A valid force of mortality function must be non-negative for all x and $\int_0^\infty \mu(x)\,dx = \infty$ because $s(\infty)$ must equal 0. All three satisfy the first condition.

(i) $\int_0^\infty BC^x\,dx = \frac{BC^x}{\ln C}\Big]_0^\infty = \infty$ ✓

(ii) $\int_0^\infty a(b+x)^{-1}\,dx = a\ln(b+x)]_0^\infty = \infty$ ✓

(iii) $\int_0^\infty (1+x)^{-3}\,dx = -\frac{1}{2}(1+x)^{-2}\Big]_0^\infty = \frac{1}{2} - 0 \neq \infty$

---

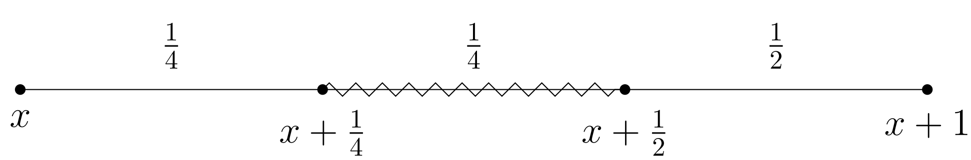
109. Under the assumption of CF ${}_tp_x = (p_x)^t$ and age doesn't matter so ${}_{1/4}p_{x+1/4} = {}_{1/4}p_x$

$${}_{1/4}p_x = \frac{49}{50}$$

$$(p_x)^{1/4} = \frac{49}{50}$$

$$p_x = \left(\frac{49}{50}\right)^4$$

For UDD we draw a picture



The probability of death is the length of the squiggly line times q_x divided by 1 minus the length of the line to the left of the squiggly line times q_x

$$ {}_{1/4}q_{x+1/4} = \frac{\frac{1}{4}q_x}{1-\frac{1}{4}q_x} = 0.0198 $$

Finally

$$ {}_{1/4}p_{x+1/4} = 1 - 0.0198 = \boxed{0.9802} $$

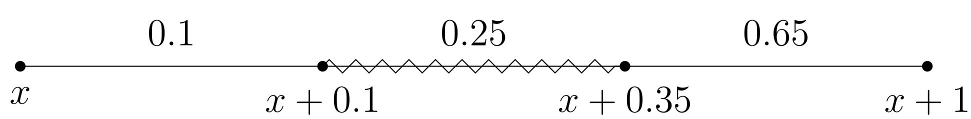
110.
$$ \int_1^\infty {}_tp_x\,\mu_{x+t}\,dt = \int_0^\infty \frac{d}{dt}\,{}_tq_x\,dt = {}_tq_x\big]_1^\infty = 1 - q_x = \boxed{p_x} $$

---

112. Under the assumption of CF ${}_tp_x = (p_x)^t$ and age doesn't matter so ${}_{0.25}p_{x+0.1} = {}_{0.25}p_x$

$$ \begin{aligned} {}_{0.25}p_{x+0.1} &= 0.95 \\ (p_x)^{0.25} &= 0.95 \\ p_x &= 0.95^4 \end{aligned} $$

For UDD we draw a picture



The probability of death is the length of the squiggly line times q_x divided by 1 minus the length of the line to the left of the squiggly line times q_x

$$ {}_{0.25}q_{x+0.1} = \frac{0.25q_x}{1-0.1q_x} = \boxed{0.04725} $$

113.

(i)
$$ \begin{aligned} {}_{t+u}q_x &\geq {}_uq_{x+t} \\ p_x - p_{x+t+u} &\geq p_{x+t} - p_{x+t+u} \\ p_x &\geq p_{x+t} \checkmark \end{aligned} $$

(ii) ${}_{u}q_{x+t} \geq {}_{t|u}q_{x+t}$

$${}_{u}q_{x+t} \geq {}_{t}p_{x}\, {}_{u}q_{x+t}$$

$$1 \geq {}_{t}p_{x} \ \checkmark$$

(iii) Let m be the median

$${}_{m}p_{x} = 0.5$$

$$\tfrac{\omega-x-m}{\omega-x} = 0.5$$

$$\omega - x - m = 0.5(\omega - x)$$

$$m = \frac{\omega - x}{2} = \mathring{e}_x \ \checkmark$$

---

114. One approach is to use integration. First note that between 0 and 1 ${}_{t}p_{x} = e^{-t\mu_x}$ and between 1 and 2 ${}_{t}p_{x} = p_x \cdot {}_{t-1}p_{x+1} = 0.98e^{-(t-1)\mu_{x+1}}$.

$$\begin{aligned}
\mathring{e}_{x:\overline{2}|} &= \int_0^2 {}_{t}p_{x}\, dt \\
&= \int_0^1 e^{-t\mu_x}\, dt + 0.98 \int_1^2 e^{-(t-1)\mu_{x+1}}\, dt \\
&= \frac{1 - e^{-t\mu_x}}{\mu_x} + 0.98 \left(\frac{1 - e^{-t\mu_{x+1}}}{\mu_{x+1}} \right) \\
&= 0.99 + 0.98(0.98) \\
&= \boxed{1.9504}
\end{aligned}$$

If you forgot how to integrate (sometimes I wish I could), then you can do the following. The $e^{-\mu(x+k)}$ column gives us p_x and p_{x+1}

$$p_x = 0.98$$

$$p_{x+1} = 0.99$$

The next column gives us $\mu(x)$ and $\mu(x+1)$

$$\frac{1 - p_x}{\mu(x)} = 0.99$$

$$\mu(x) = \frac{2}{99}$$

$$\frac{1 - p_{x+1}}{\mu(x+1)} = 0.98$$

$$\mu(x+1) = \frac{4}{98}$$

Now we need $a(x)$ and $a(x+1)$

$$\begin{aligned} a(x) &= \frac{L_x - \ell_{x+1}}{\ell_x - \ell_{x+1}} \\ &= \frac{\frac{L_x}{\ell_x} - \frac{\ell_{x+1}}{\ell_x}}{\frac{\ell_x - \ell_{x+1}}{\ell_x}} \\ &= \frac{\frac{\frac{d_x}{m_x}}{\ell_x} - p}{q} \\ &= \frac{\frac{q}{\mu} - p}{q} \end{aligned}$$

Using this we get

$$a(x) = \frac{\frac{0.02}{\frac{2}{99}} - 0.98}{0.02} = 0.5$$

$$a(x+1) = \frac{\frac{0.04}{\frac{4}{98}} - 0.96}{0.04} = 0.5$$

Finally

$$\begin{aligned} \mathring{e}_{x:\overline{2}|} &= q_x(0.5) + {}_{1|}q_x(1.5) + {}_2p_x(2) \\ &= 0.02(0.5) + 0.98(0.04)(0.5) + 0.98(0.96)(0.5) \\ &= \boxed{1.9504} \end{aligned}$$

116.

$$\begin{aligned} \ell_2 &= \ell_1\, p_1 = 9700(1 - .02) = 9506 \\ \ell_3 &= \ell_2\, p_2 = 9506(1 - .02) = 9315.88 \\ \ell_4 &= \ell_3 - d_3 = 9315.88 - 232 = 9083.88 \end{aligned}$$

$$\ell_5 = \ell_4\, p_4 = 9083.88(1 - .026) = \boxed{8847.70}$$

---

117. The average lifetime lived between ages x and $x+1$ by those who die between ages x and $x+1$ is $a(x)$ and under CF we can calculate this as

$$\begin{aligned} a(x) &= \frac{L_x - \ell_{x+1}}{\ell_x - \ell_{x+1}} \\ &= \frac{\frac{L_x}{\ell_x} - \frac{\ell_{x+1}}{\ell_x}}{\frac{\ell_x - \ell_{x+1}}{\ell_x}} \\ &= \frac{\frac{\frac{d_x}{m_x}}{\ell_x} - p}{q} \\ &= \frac{\frac{q}{\mu} - p}{q} \end{aligned}$$

Now we find p, q and μ between ages 1 and 2

$$\begin{aligned} p &= \frac{s(2)}{s(1)} = \frac{8^2}{9^2} = \frac{64}{81} \\ q &= 1 - \frac{64}{81} = \frac{17}{81} \\ \mu &= -\ln\left(\frac{64}{81}\right) \end{aligned}$$

Now we plug those into the first equation

$$a(1) = \frac{\frac{\frac{17}{81}}{-\ln\left(\frac{64}{81}\right)} - \frac{64}{81}}{\frac{17}{81}} = \boxed{0.4804}$$

---

118. We are given MDML with $\omega = 10$ and $a = 2$. Under MDML

$$\begin{aligned} \mu(x) &= \frac{a}{\omega - x} \\ q_x &= 1 - p_x = 1 - \left(\frac{\omega - x - 1}{\omega - x}\right)^a \end{aligned}$$

Thus

$$\mu(1) - q_1 = \frac{2}{10-1} - \left(1 - \left(\frac{8}{9}\right)^2\right) = \boxed{0.0123}$$

119. We are given MDML with $\omega = 10$ and $a = 2$.

$$\begin{aligned} \overset{\circ}{e}_{1:\overline{1|}} &= p_1 + q_1\, a(1) \\ \overset{\circ}{e}_1 - p_1\ \overset{\circ}{e}_2\ &= p_1 + q_1\, a(1) \\ \frac{9}{3} - \left(\frac{8}{9}\right)^2 \frac{8}{3} &= \left(\frac{8}{9}\right)^2 + \left(1 - \left(\frac{8}{9}\right)^2\right) a(1) \\ a(1) &= \boxed{0.4902} \end{aligned}$$

---

120. We are given MDML with $\omega = 10$ and $a = 2$. Under MDML $\overset{\circ}{e}_x = \frac{\omega - x}{a+1}$

$$\overset{\circ}{e}_1 = \frac{9}{3} = \boxed{3}$$

---

121. One approach is to build a table

x	ℓ_x	${}_5d_x$
20	1000 (arbitrary)	
25		
30		
35		
40	ℓ_{40}	$1000(.04) = 40$
45		$\ell_{40}(.0475)$
50	$1000(0.7) = 700$	

There are two ways to arrive at ℓ_45

$$\begin{aligned} \ell_{40} - 40 &= 700 + \ell_{40}(.0475) \\ \ell_{40} &= 777 \\ \ell_{45} &= 777 - 40 = 737 \end{aligned}$$

Finally

$${}_{45}p_{20} = \frac{737}{1000} = \boxed{0.737}$$

---

124. e_x is not in any of the answer choices so let's rewrite that in terms of e_{x+1}

$$e_{x+1} > e_x$$

$$e_{x+1} > p_x\,(1+e_{x+1})$$
$$e_{x+1}\,(1-p_x) > p_x$$
$$e_{x+1} > \boxed{\tfrac{p_x}{q_x}}$$

---

126. Under CF age doesn't matter thus

$$
\begin{aligned}
{}_{3|3}q_{33} &= {}_3p_{33}\cdot{}_3q_{36} = {}_3p_{33}\cdot{}_3q_{33}\\
0.003 &= {}_3p_{33}(1-{}_3p_{33})\\
{}_3p_{33}^2 - {}_3p_{33} + 0.003 &= 0 \text{ use quadratic formula}\\
{}_3p_{33} &= \frac{-(-1)\pm\sqrt{(-1)^2-4\cdot 1\cdot 0.003}}{2\cdot 1} = 0.996991 \text{ or } 0.003009054\\
e^{-3\mu} &= 0.996991 \text{ or } 0.003009054\\
\mu &= \frac{\ln(0.996991)}{-3} \text{ or } \frac{\ln(0.003009054)}{0.003}\\
&= 1.0045 \text{ or } 0.00100453,\ \text{ but } 1.045 > 1 \text{ so it is not allowed.}\\
\mu &= .00100453\\
1000\mu &\approx \boxed{1.0}
\end{aligned}
$$

---

127. Complete the table

x	ℓ_x	q_x	d_x
50	1000	0.02	$1000(.02) = 20$
51	$1000 - 20 = 980$		32
52	$980 - 32 = 948$		30
53	$948 - 30 = 918$		28
54	$918 - 28 = 890$	0.028	$890(.028) = 25$

$$800\,{}_{4|}q_{50} = 800\frac{25}{1000} = \boxed{20}$$

132. The adjusted values are

$$\begin{aligned} q'_x &= 1-(1-0.07)^{1.1} = 0.077 \\ q'_{x+1} &= 1-(1-0.10)^{1.05} = 0.105 \end{aligned}$$

Now we calculate the curtate expectancy

$$\begin{aligned} e_x &= p_x + {}_2p_x + {}_3p_x\,(1+e_{x+3}) \\ &= 0.93 + 0.93(0.9) + 0.93(0.9)(0.89)(1+5) \\ &= 6.237 \end{aligned}$$

$$\begin{aligned} e'_x &= p'_x + {}_2p'_x + {}_3p'_x\,(1+e_{x+3}) \\ &= 0.923 + 0.923(0.895) + 0.923(0.895)(0.89)(1+5) \\ &= 6.160 \end{aligned}$$

The difference is

$$e_x - e'_x = 6.237 - 6.160 = \boxed{0.077}$$

133.

$$\begin{aligned} {}_{20}p_{60} &= 0.1 \\ e^{-20\mu} &= 0.1 \\ \mu &= \boxed{0.115} \end{aligned}$$

---

134. Set $\ell_{70} = 100$ (arbitrary). We are given:

(i) $\ell_{90} = \ell_{70}\sqrt{0.16} = 100(0.4) = 40$

(ii) $\ell_{100} = \ell_{80}\sqrt{0.01} = 0.1\,\ell_{80}$

(iii) $\ell_{100} = \ell_{70}(0.08) = 100(0.08) = 8$

Combining 2 and 3 we have

$$\begin{aligned} 0.1\,\ell_{80} &= 8 \\ \ell_{80} &= 80 \end{aligned}$$

Finally

$$_{10}p_{80} = \frac{\ell_{90}}{\ell_{80}} = \frac{40}{80} = \boxed{0.5}$$

---

137. Completing the table we have

x	ℓ_x	d_x	$_{x\|}q_0$
0	1,000,000	200,000	$\frac{200{,}000}{1{,}000{,}000} = 0.2$
1	800,000	200,000	$\frac{200{,}000}{1{,}000{,}000} = 0.2$
2	600,000	300,000	$\frac{300{,}000}{1{,}000{,}000} = 0.3$
3	300,000	300,000	$\frac{300{,}000}{1{,}000{,}000} = 0.3$

So of the original 2,500 bulbs 20% burn out during first year, 20% during second year and 30% during the third year.

Year 1	Year 2	Year 3
2,500(0.2) = 500	2,500(0.2) = 500	2,500(0.3) = 750

Now we need to account for the 500 bulbs that burned out during the first year. 20% of these burn out during year 2 (one year later) and 20% burn out during year 3 (two years later)

Year 1	Year 2	Year 3
2,500(0.2) = 500	2,500(0.2) = 500 500(0.2) = 100	2,500(0.3) = 750 500(0.2) = 100

Now we account for the 500 + 100 = 600 bulbs that burned out during year 2. 20% of those will burn out during the third year (one year later). So our final table looks like

Year 1	Year 2	Year 3
2,500(0.2) = 500	2,500(0.2) = 500 500(0.2) = 100	2,500(0.3) = 750 500(0.2) = 100 600(0.2) = 120

So the total number of bulbs replaced during the 3rd year = 750+100+120 = 970.

138. The future lifetime of (50) is uniformly distributed from 0 to $\omega - 50$

$$\text{Var}[T(50)] = 192$$

$$\frac{(\omega - 50)^2}{12} = 192$$

$$\omega = 98$$

---

139. Use $\overset{\circ}{e}_0 = 25$ to find ω

$$\overset{\circ}{e}_0 \ = 0$$

$$\frac{\omega - 0}{2} = 25$$

$$\omega = 50$$

The future lifetime of (10) is uniformly distributed from 0 to 40

$$\text{Var}[T(10)] = \frac{(40-0)^2}{12} = 133$$

---

140. Key facts

$$\ell_{x+t} = \ell_x - t \cdot d_x$$

$$\mu(x+t) = \frac{-\frac{d}{dt}\ell_{x+t}}{\ell_{x+t}} = \frac{d_x}{\ell_x - t \cdot d_x}$$

If we let $\ell_{80} = 100$ then

$$\mu(80.5) = 0.0202$$

$$\frac{d_{80}}{100 - 0.5d_{80}} = 0.0202$$

$$d_{80} = 2$$

$$\ell_{81} = 100 - 2 = 98$$

Do the same for $\mu(81.5)$

$$\mu(81.5) = 0.0408$$

$$\begin{aligned}
\frac{d_{81}}{98 - 0.5 d_{81}} &= 0.0408 \\
d_{81} &= 3.9185 \\
\ell_{82} &= 98 - 3.9185 = 94.0815
\end{aligned}$$

Likewise we find $d_{82} = 5.6488$ and $\ell_{83} = 88.4327$.

$$\begin{aligned}
\ell_{80.5} &= \frac{100 + 98}{2} = 99 \\
\ell_{82.5} &= \frac{94.0815 + 88.4327}{2} = 91.2571
\end{aligned}$$

Finally

$${}_{2}q_{80.5} = \frac{\ell_{82.5} - \ell_{80.5}}{\ell_{80.5}} = \frac{99 - 91.2571}{99} = \boxed{0.0782}$$

---

141. If θ didn't vary then ${}_{0.5}p_0 = e^{-0.5\theta}$, but since it does we want

$$\begin{aligned}
\mathrm{E}(e^{-0.5\theta}) &= \int_1^{11} e^{-0.5\theta} \cdot f(\theta)\, d\theta \\
&= \int_1^{11} e^{-0.5\theta} \left(\frac{1}{10}\right) d\theta \\
&= -\frac{1}{5} e^{-0.5\theta} \Big]_1^{11} \\
&= \frac{1}{5}\left(e^{-0.5} - e^{-5.5}\right) \\
&= \boxed{0.12}
\end{aligned}$$

142.

$$\begin{aligned}
(1.16)^{20} \cdot v^{20} \cdot {}_{20}p_0 &= (1.10)^{20} \cdot v^{20} \\
(1.16)^{20}\, e^{-20\mu} &= (1.10)^{20} \\
\mu &= 0.05312
\end{aligned}$$

Let m be the median then

$${}_{m}p_0 = 0.5$$

$$e^{-m\mu} = 0.5$$
$$m = \boxed{13.0487}$$

143.

$$_{12|}q_{20} \cdot {}_{35}p_{21} = \left(\frac{\ell_{32} - \ell_{33}}{\ell_{20}}\right)\left(\frac{\ell_{56}}{\ell_{21}}\right) = \boxed{0.001489}$$

---

144. No air ballooning

$$\begin{aligned}
\mathring{e}_{25:\overline{11}|} &= {}_{11}p_{25}(11) + {}_{11}q_{25}(5.5) \\
&= \tfrac{64}{75}(11) + \tfrac{11}{75}(5.5) \\
&= 10.193
\end{aligned}$$

With air ballooning

$$\begin{aligned}
\mathring{e}_{25:\overline{11}|} &= \mathring{e}^{\text{CF}}_{25:\overline{1}|} + p_{25}\ \mathring{e}^{\text{DML}}_{26:\overline{10}|} \\
&= \mathring{e}_{25}\ (1 - p_{25}) + p_{25}\ \mathring{e}_{26:\overline{10}|} \\
&= \tfrac{1}{0.1}\left(1 - e^{-0.1}\right) + e^{-0.1}\left[\tfrac{64}{74}(10) + \tfrac{10}{74}(5)\right] \\
&= 9.389
\end{aligned}$$

The difference is

$$10.193 - 9.389 = \boxed{0.804}$$

146.

$$\begin{aligned}
\frac{e^N_{25}}{e^M_{25}} &= \frac{p^N_{25}\ \left(1 + e^M_{26}\right)}{p^M_{25}\ \left(1 + e^M_{26}\right)} \\
&= \frac{\exp\left(-\int_0^1 \mu^M_{25}(t) + 0.1(1-t)\, dt\right)}{\exp\left(-\int_0^1 \mu^M_{25}(t)\, dt\right)} \\
&= \exp\left(-\int_0^1 0.1(1-t)\, dt\right)
\end{aligned}$$

$$= \exp\left(\frac{0.1}{2}(1-t)^2\Big]_0^1\right)$$

$$= \exp\left(-\frac{0.1}{2}\right)$$

$$e_{25}^N = 10\exp\left(-\frac{0.1}{2}\right) = \boxed{9.5123}$$

---

147. First find ℓ_{21}

$$\ell_{21} = \tfrac{4}{5}\ell_{20} + \tfrac{1}{5}\ell_{25} = 46{,}267$$

Now calculate the total time hospitalized for each range. If you survive to the end of the range, then it is easy you get all the days in the range. If you die before the end of the range, then on average you get half of the range since we assume UDD.

Range	L_{Range}
21-25	$4(17{,}384) + 2(46{,}267 - 17{,}384) = 127{,}302$
25-30	$5(5{,}349) + 2.5(17{,}384 - 5{,}349) = 56{,}832.50$
30-35	$5(1{,}337) + 2.5(5{,}349 - 1{,}337) = 16{,}715$
35-40	$5(0) + 2.5(1{,}337 - 0) = 3{,}342.50$
Total	204,192

Now we simply divide by ℓ_{21} to get the average

$$\frac{204{,}192}{46{,}267} = \boxed{4.41}$$

148.

$${}_{0.4}p_0 = \exp\left(-\int_0^{0.4} \mu(x)\,dx\right)$$

$$0.5 = \exp\left(-\int_0^{0.4} F + e^{2x}\,dx\right)$$

$$0.5 = \exp(-0.4F)\,\exp\left(-\frac{1}{2}\,e^{2x}\Big]_0^{0.4}\right)$$

$$0.5 = \exp(-0.4F)\,\exp\left(\frac{1}{2}\left(1 - e^{0.8}\right)\right)$$

$$F = \boxed{0.2}$$

---

149. Before breakthrough

$$\overset{\circ}{e}_{30} = \frac{70}{2} = 35$$

After breakthrough

$$\begin{aligned}
\overset{\circ}{e}_{30} &= 35 + 4 \\
\frac{\omega - 30}{2} &= 39 \\
\omega &= \boxed{108}
\end{aligned}$$

---

151. This is the curve of deaths $\boxed{\ell_x \, \mu(x)}$

---

152. Under UDD $\ell_{96.5}$ is just the arithmetic mean of ℓ_{96} and ℓ_{97}

$$\begin{aligned}
\ell_{96.5} &= \frac{\ell_{96} + \ell_{97}}{2} \\
480 &= \frac{600 + \ell_{97}}{2} \\
\ell_{97} &= 360
\end{aligned}$$

We use the same idea to find $\ell_{98} = 216$, then we use those values to find a revised value for $\ell_{97.5}$ under the assumption of constant force

$$\begin{aligned}
p_{97} &= \frac{216}{360} \\
{}_{0.5}p_{97} &= \left(\frac{216}{360}\right)^{0.5} \\
\ell_{97.5} &= \ell_{97}\left({}_{0.5}p_{97}\right) = 360\left(\frac{216}{360}\right)^{0.5} = \boxed{278.85}
\end{aligned}$$

---

153. Since μ is a random variable, what we really want is

$$\mathrm{E}[e^{-\mu}] = \int_0^2 e^{-\mu}\left(\tfrac{1}{2}\right) d\mu$$

$$= -\tfrac{1}{2}e^{-\mu}\Big]_0^2$$
$$= \tfrac{1}{2}(1 - e^{-2}) = 0.432$$
$$q_x = 1 - 0.432 = \boxed{0.568}$$

---

154. First build a table

x	ℓ_x	d_x
1	200	20
2	180	9
3	171	

Since dropouts are UDD

$$\ell_{2.5} = \frac{180 + 171}{2} = 175.5$$
$${}_{0.5}d_2 = 180 - 175.5 = 4.5$$

Finally

$$\overset{\circ}{e}_{1:\overline{1.5}|} = \frac{20(0.5) + 4.5(1.25) + 175.5(1.5)}{200} = \boxed{1.394}$$

---

155. We need to split the expectation at age 40 since that is where the force of mortality changes

$$\overset{\circ}{e}_{25:\overline{25}|} = \overset{\circ}{e}_{25:\overline{15}|} + {}_{15}p_{25}\ \overset{\circ}{e}_{40:\overline{10}|}$$
$$= \overset{\circ}{e}_{25}\ (1 - {}_{15}p_{25}) + {}_{15}p_{25}\ \overset{\circ}{e}_{40}\ (1 - {}_{10}p_{40})$$
$$= \tfrac{1}{0.04}\left(1 - e^{-15(0.04)}\right) + e^{-15(0.04)}\left(\tfrac{1}{0.05}\right)\left(1 - e^{-10(0.05)}\right)$$
$$= \boxed{15.6}$$

---

157. First we find α

$$\mathrm{E}[X] = 8$$
$$\frac{\alpha(4)}{\alpha - 1} = 8$$

$$\alpha = 2$$

Then we need to find $s(6)$

$$s(x) = 1 - F(x)$$
$$s(x) = 1 - \left(1 - \left(\frac{4}{x}\right)^2\right) = \left(\frac{4}{x}\right)^2$$
$$s(6) = \left(\frac{4}{6}\right)^2 = \boxed{0.444}$$

158.

$$\mu(x) = \frac{-s'(x)}{s(x)} = -\frac{-\frac{e^x}{100}}{1 - \frac{e^x}{100}}$$
$$\mu(4) = \frac{\frac{e^4}{100}}{1 - \frac{e^4}{100}} = \boxed{1.20}$$

---

159. First note that R is just the probability of death between ages x and $x+1$, i.e. q_x. S is an adjusted q_x.

$$S = 1 - exp\left(-\int_0^1 \mu_x(t)dt\right) \cdot exp\left(-\int_0^1 kdt\right)$$
$$= 1 - p_x\, e^{-k}$$

We are given $S = 0.75R$, therefore

$$1 - p_x\, e^{-k} = 0.75\, q_x$$
$$e^{-k} = \frac{1 - 0.75q_x}{p_x}$$
$$k = \ln\left[\frac{p_x}{1 - 0.75q_x}\right]$$
$$k = \boxed{\ln\left[\frac{1 - q_x}{1 - 0.75q_x}\right]}$$

161. The sum of a pdf over all possible values must equal 1

$$\begin{aligned}
1 &= \int_0^{100} f_T(t)dt \\
1 &= \int_0^{50} kf_1(t)dt + \int_{50}^{100} 1.2\left(\tfrac{1}{100}\right)dt \\
1 &= k\left[F_1(50) - F_1(0)\right] + 1.2\left(\tfrac{50}{100}\right) \\
0.4 &= k\left({}_{50}q_0\right) \\
k &= 0.4 \cdot \frac{\ell_0}{\ell_0 - \ell_{50}} \\
&= 0.4 \cdot \frac{10m}{10m - 8950901} = 3.8128
\end{aligned}$$

Where F_1 is the cdf of f_1. Next we find ${}_{10}p_{40}$

$$\begin{aligned}
{}_{10}p_{40} &= \frac{s(50)}{s(40)} \\
&= \frac{1 - F_T(50)}{1 - F_T(40)} \\
&= \frac{1 - k_{50}q_0}{1 - k_{40}q_0}
\end{aligned}$$

Where F_T is the cdf of f_T. We already have $k_{50}q_0$, but we need $k_{40}q_0$

$$k_{40}q_0 = 3.8128 \cdot \frac{10m - 9313166}{10m} = 0.262$$

Plugging that back into the equation for ${}_{10}p_{40}$

$${}_{10}p_{40} = \frac{1 - 0.4}{1 - 0.262} = \boxed{0.813}$$

162. The 75th percentile of the future lifetime is the age where the probability of death before that age is 75% or the probability of survival to that age is $1 - 75\% = 25\%$

$$0.25 = {}_hp_x$$

$$0.25 = 0.3e^{-0.2h} + 0.7e^{-0.1h}$$
$$0 = 0.3\left(e^{-0.1h}\right)^2 + 0.7e^{-0.1h} - 0.25$$
$$e^{-0.1h} = \frac{-0.7 \pm \sqrt{0.7^2 - 4(0.3)(-0.25)}}{2(0.3)}$$
$$= 0.31469907 \qquad \text{(throw out negative solution)}$$
$$h = \boxed{11.56138}$$

---

163. Just list all the possible values for $K \wedge 3$ and the corresponding probability

K	$K \wedge 3$	Prob
0	0	0.1
1	1	$0.9(0.2) = 0.18$
2	2	$0.9(0.8)(0.3) = 0.216$
3+	3	$1 - (0.1 + 0.18 + 0.216) = 0.504$

$$\mathrm{E}[K \wedge 3] = 1(0.18) + 2(0.216) + 3(0.504) = 2.124$$
$$\mathrm{E}[(K \wedge 3)^2] = 1^2(0.18) + 2^2(0.216) + 3^2(0.504) = 5.58$$
$$\mathrm{Var}[K \wedge 3] = 5.58 - 2.124^2 = \boxed{1.069}$$

---

164. We use recursion

$$\mathring{e}_0 = \mathring{e}_{0:\overline{40}|} + {}_{40}p_0 \ \mathring{e}_{40}$$
$$62 = \left(40 - 0.005(40)^2\right) + 0.6 \ \mathring{e}_{40}$$
$$\mathring{e}_{40} = \boxed{50}$$

---

166. This is MDML with $\omega = 100$ and $a = 2$

$${}_{10|}q_{65} = {}_{10}p_{65} \cdot q_{75}$$
$$= \left(\frac{25}{35}\right)^2 \cdot \left(1 - \left(\frac{24}{25}\right)^2\right)$$

$$= \boxed{\frac{1}{25}}$$

---

167. Find ω

$$\begin{aligned} \mathring{e}_{20} &= 30 \\ \frac{\omega - 20}{2} &= 30 \\ \omega &= 80 \end{aligned}$$

Thus

$$q_{20} = \boxed{\frac{1}{60}}$$

---

168. We need to find where 25% have died

$$\begin{aligned} {}_{y}q_{30} &= 0.25 \\ {}_{y}p_{30} &= 0.75 \\ \frac{\ell_{y+30}}{\ell_{30}} &= 0.75 \\ \ell_{y+30} &= 9{,}501{,}381(0.75) = 7{,}126{,}035.75 \end{aligned}$$

From the table we see that

$$\ell_{67} = 7{,}201{,}635 \text{ and } \ell_{68} = 7{,}018{,}432$$

Using linear interpolation

$$\begin{aligned} 7{,}126{,}035.75 &= 7{,}201{,}635(1-t) + 7{,}018{,}432t \\ t &= 0.41265 \end{aligned}$$

Thus, the fund is disolved in $67.41265 - 30 = 37.41265$ years

$$\begin{aligned} 4000P(1.12)^{37.41265} &= 3000(50000) \\ P &= \boxed{540.32} \end{aligned}$$

169. Key fact

$$q_{60} = \frac{s(60) - s(61)}{s(60)} = \boxed{0.081}$$

Where

$$s(60) = \tfrac{1}{2}e^{-60(0.10)} + \tfrac{1}{2}e^{-60(0.08)} = 0.00535425$$
$$s(61) = \tfrac{1}{2}e^{-61(0.10)} + \tfrac{1}{2}e^{-61(0.08)} = 0.00491994$$

170. We find $\mathring{e}_{20}$ using integration since this isn't one of our known mortality laws

$$\mathring{e}_{20} = \int_0^\infty {}_tp_{20}\,dt$$

We can find ${}_tp_{20}$ as follows

$${}_tp_{20} = \frac{s(20+t)}{s(20)}$$
$$s(x) = 1 - F(x) = \left(\frac{50}{50+x}\right)^3$$
$${}_tp_{20} = \left(\frac{70}{70+t}\right)^3$$

Finally

$$\begin{aligned}\mathring{e}_{20} &= \int_0^\infty \left(\frac{70}{70+t}\right)^3 dt \\ &= 70^3 \int_0^\infty (70+t)^{-3} dt \\ &= \boxed{35}\end{aligned}$$

171. We have MDML with $\omega = 100$ and $a = \frac{1}{2}$

$${}_{15|13}q_{36} = {}_{15}p_{36} - {}_{28}p_{36}$$

$$= \left(\frac{49}{64}\right)^{\frac{1}{2}} - \left(\frac{36}{64}\right)^{\frac{1}{2}}$$
$$= \boxed{0.125}$$