Force of Mortality (aka hazard rate)

AM: 3.1 – 3.3
MQR: 3.1.3 – 3.1.4, 3.3.3 – 3.3.4, 4.3.1 – 4.3.2

Interest Theory

$$i_t = \frac{a(t) - a(t-1)}{a(t-1)} \qquad q_x = \frac{s(x) - s(x+1)}{s(x)}$$

$$\delta_t = \frac{\frac{d}{dt} a(t)}{a(t)} \qquad \mu_x = \frac{-\frac{d}{dx} s(x)}{s(x)}$$

$$\mu(x) = \frac{-\frac{d}{dx}\left(1 - F(x)\right)}{s(x)} = \frac{f(x)}{s(x)}$$

Probability Density Function, $f_x(x)$

$$f(x) \geq 0$$

$$\int_0^\infty f(x)\,dx = 1$$

$$\mu(x) = \frac{f(x)}{s(x)}$$

$$f(x) = s(x)\,\mu(x)$$

prob age at death is x =

prob survive to age x and then hit by force of mort.

$$f(x) = \frac{l_x}{l_0} \cdot \mu(x)$$

$l_x \mu(x)$ curve of deaths

Curve of Deaths
(proportional to pdf)

$$\mu(x) = -\frac{s'(x)}{s(x)}$$

$$\mu(x+t) = \mu_x(t) = -\frac{s'(x+t)}{s(x+t)} = -\frac{d}{dt}\ln s(x+t)$$

$$-\int_0^n \mu(x+t)\,dt = \ln s(x+n) - \ln s(x)$$

$$= \ln\left(\frac{s(x+n)}{s(x)}\right) = \ln {}_nP_x$$

$$\boxed{e^{-\int_0^n \mu(x+t)\,dt} = {}_nP_x}$$

$$s(n) = {}_nP_0 = e^{-\int_0^n \mu_t\,dt} \qquad a(n) = e^{\int_0^n \delta_t\,dt}$$

<u>Properties for $\mu(x)$</u>

1. $\mu(x) \geq 0$
2. $\int_0^\infty \mu(x)\,dx = \infty$ since ${}_\infty P_0 = 0$

$y = x + t$

$${}_nP_x = e^{-\int_x^{x+n} \mu(x)\,dx}$$

<u>Examples</u>

$$e^{-\int_0^{15} \mu(y)\,dy} = {}_{15}P_0$$

$$e^{-\int_0^{15} \mu(20+y)\,dy} = e^{-\int_0^{15} \mu_{20}(y)\,dy} = {}_{15}P_{20}$$

$$e^{-\int_5^{25} \mu(y)\,dy} = {}_{20}P_5$$

$$\frac{e^{-\int_0^{10} \mu(40+t)\,dt}}{e^{-\int_0^{5} \mu(40+t)\,dt}} = \frac{{}_{10}P_{40}}{{}_5P_{40}} = {}_5P_{45}$$

$${}_{10}P_{40} = {}_5P_{40} \cdot {}_5P_{45}$$

$$\mu(x) = \frac{f(x)}{s(x)}$$

$$f_X(x) = s(x)\,\mu(x)$$

$$= {}_xp_0\,\mu(x)$$

$$f_{T(x)}(t) = \frac{d}{dt} F_{T(x)}(t) = \frac{d}{dt}\,{}_tq_x$$

$$= \frac{d}{dt}\left[1 - \frac{s(x+t)}{s(x)}\right]$$

$$= \frac{-s'(x+t)}{s(x)} \cdot \frac{s(x+t)}{s(x+t)}$$

$$= \frac{s(x+t)}{s(x)} \cdot \left[-\frac{s'(x+t)}{s(x+t)}\right]$$

$$= {}_tp_x \cdot \mu(x+t)$$

$\therefore$ ${}_tp_x\,\mu(x+t)$ is probability that (x) dies b/w t and $t+dt$

$$\int_0^\infty {}_tp_x \, \mu(x+t)\, dt = 1$$

$$\int_0^n {}_tp_x \, \mu(x+t)\, dt = {}_nq_x$$

$$\int_n^\infty {}_tp_x \, \mu(x+t)\, dt = {}_np_x$$

$$\int_u^{u+s} {}_tp_x \, \mu(x+t)\, dt = {}_{u|s}q_x$$

$$\frac{d}{dt}\, {}_tq_x = {}_tp_x \, \mu(x+t) \quad \star$$

$$\frac{d}{dt}\, {}_tp_x = \frac{d}{dt}\left(1 - {}_tq_x\right) = -{}_tp_x \, \mu(x+t)$$

$$\boxed{\begin{aligned} \mu(x+t) &= \frac{-s'(x+t)}{s(x+t)} = \frac{-\frac{d}{dx}\, l_{x+t}}{l_{x+t}} = \frac{-\frac{d}{dt}\, l_{x+t}}{l_{x+t}} \\ &= \frac{\frac{d}{dt}\, {}_tq_x}{{}_tp_x} = \frac{-\frac{d}{dt}\, {}_tp_x}{{}_tp_x} \end{aligned}}$$

$$\frac{d}{dx}\, l_{x+t} = \frac{d}{dx}\, l_0\, s(x+t) = \frac{d}{dt}\, l_0\, s(x+t)$$

$$= l_0\, s'(x+t)$$

$$\boxed{\mu(x+t) = \frac{-s'(x+t)}{s(x+t)}}$$

$$= -l_0\, s(x+t)\, \mu(x+t)$$

$$= -\, l_{x+t}\, \mu(x+t)$$

$$(ab)' = a' \cdot b + a \cdot b'$$

$$\frac{d}{dx}\, {}_tp_x = \frac{d}{dx}\, \frac{s(x+t)}{s(x)} = \frac{s'(x+t)}{s(x)} + s(x+t)\left[-\frac{1}{s(x)^2} \cdot s'(x)\right]$$

$$= \frac{s(x+t)}{s(x)} \cdot \frac{s'(x+t)}{s(x+t)} + \frac{s(x+t)}{s(x)} \cdot \left[-\frac{s'(x)}{s(x)}\right]$$

$$= {}_tp_x\left(-\mu(x+t)\right) + {}_tp_x\, \mu(x)$$

$$= {}_tp_x\left(\mu(x) - \mu(x+t)\right)$$

$$\star \quad \frac{d}{dt}\, {}_tp_x = -\, {}_tp_x\, \mu(x+t)$$

$$\mu(x) = \frac{-\frac{d}{dx} l_x}{l_x}$$

$$\mu(x) = \frac{1}{80-x} \qquad \text{find } s(20)$$

$$l_x = 80 - x$$

$$-\frac{d}{dx} l_x = 1$$

$$s(20) = \frac{l_{20}}{l_0} = \frac{60}{80}$$

$$\mu(x) = \frac{3x^2}{1000-x^3} \qquad l_x = 1000 - x^3$$

$$s(x) = \frac{l_x}{l_0} = \frac{1000-x^3}{1000}$$

$$\mu(x) = \frac{x^2}{1000-x^3} = \frac{1}{3}\left[\frac{3x^2}{1000-x^3}\right]$$

$$l_x = (1000-x^3)^{\frac{1}{3}}$$

$$\mu(x) = \frac{1}{240-2x} \qquad \text{find } {}_{30}q_{20}$$

$$= \frac{1}{2}\left(\frac{2}{240-2x}\right)$$

$$l_x = (240-2x)^{\frac{1}{2}}$$

$${}_{30}q_{20} = 1 - {}_{30}p_{20}$$

$$= 1 - \frac{l_{50}}{l_{20}}$$

$$= 1 - \frac{\sqrt{140}}{\sqrt{200}} = 1 - \frac{2\sqrt{70}}{2\sqrt{100}}$$

$$= \frac{10-\sqrt{70}}{10}$$

$$\mu^*(x+t) = K\mu(x+t) \qquad t \geq 0$$

$$_nP_x^* = e^{-\int_0^n \mu^*(x+t)\,dt}$$

$$= e^{-\int_0^n K\mu(x+t)\,dt}$$

$$= \left[e^{-\int_0^n \mu(x+t)\,dt}\right]^K$$

$$= \left[_nP_x\right]^K$$

If $\mu^*(x+t) = K\mu(x+t)$, then $_nP_x^* = (_nP_x)^K$

Given $\mu(x+t)$: $_5q_{40} = 0.04$

If $\mu^*(x+t) = 2\mu(x+t)$, find $_5q_{40}^*$

$$_5q_{40}^* = 1 - {_5P_{40}^*} = 1 - .96^2$$

$$= .0784 \neq .04^2$$

$\mu(x) = 0.04x$ find ${}_nP_{10}$

$${}_nP_{10} = e^{-\int_{10}^{10+n} 0.04x\,dx} = e^{-0.02x^2\Big]_{10}^{10+n}}$$

$$= e^{-0.02(n^2+20n)}$$

Integrals

2, 33, 34, 46

$$\int_0^\infty l_x\,\mu(x)\,dx = l_0$$

$$\int_0^\infty l_{x+t}\,\mu(x+t)\,dt = l_x$$

$$\int_0^n l_{x+t}\,\mu(x+t)\,dt = l_x - l_{x+n}$$

$$\int_n^\infty l_{x+t}\,\mu(x+t)\,dt = l_{x+n}$$

$$\frac{\int_{20}^{30} l_{50+t}\,\mu(50+t)\,dt}{l_{50}} = \frac{l_{20} - l_{80}}{l_{50}}$$

$$\int_{20}^{30} {}_tp_{50}\,\mu(50+t)\,dt = {}_{20|10}q_{50}$$