Maximum Likelihood Estimates I

IMS: 5.2

Example 1: Suppose claim sizes are exponential $f_\lambda(x) = \lambda e^{-\lambda \cdot x}$

λ is unknown

Data: first 5 claims are:

5.5, 8.0, 1.7, 2.7, and 10.7

Question: what is our "best" estimate of λ?

Answer: a single number, e.g., .23, is called a Point Estimate

Example 0: The number of accidents per day has a Poisson distribution. If there were 3 accidents on Tuesday, which is "more likely", that the expected number of accidents per day is 2, or that it is 3?

Estimate the expected number of accidents per day.

θ = expected # of accidents

X = actual # on Tuesday

$$p_\theta(x) = p(x;\theta) = e^{-\theta}\frac{\theta^x}{x!}$$

Assume that before looking at data, each value of θ is equally likely

$$P[\theta=2 \mid X=3] = \frac{P[\theta=2, X=3]}{P[X=3]}$$

$$= \frac{\frac{1}{2}}{P[X=3]} \cdot e^{-2}\frac{2^3}{3!}$$

$$P[\theta=3 \mid X=3] = \frac{P[\theta=3, X=3]}{P[X=3]} = \frac{P[\theta=3]\,p(3;3)}{P[X=3]}$$

$$= \frac{\frac{1}{2}}{P[X=3]} \cdot e^{-3}\frac{3^3}{3!}$$

$$P[\theta=2|X=3] = c\,\frac{e^{-2}2^3}{3!}$$

$$P[\theta=3|X=3] = c\cdot \underbrace{\frac{e^{-3}3^3}{3!}}_{L(\theta)}$$

$$L(\theta) = L(\theta;x) = f(x;\theta)$$

$$= e^{-\theta}\frac{\theta^x}{x!} = \text{"likelihood"}$$

$$L(2) = \frac{e^{-2}2^3}{3!} = .18$$

$$L(3) = .22 > L(2)$$

so 3 is "more likely" than 2.

Note: likelihoods do not need sum to 1.

MLE = maximum likelihood estimate

= θ that maximizes $L(\theta)$

Want to max $L(\theta) = e^{-\theta} \dfrac{\theta^x}{x!}$

Trick: $\ell(\theta) =$ log-likelihood

$= \ln L(\theta)$

$$\ell(\theta) = -\theta + x \ln \theta - \ln x!$$

$\ln(t)$ is increasing

so if t max's $L(\theta)$,

it max $\ell(\theta)$

Point: need to maximize $\ell(\theta)$

$$0 = \frac{d}{d\theta} \ell(\theta) = -1 + \frac{x}{\theta} + 0$$

$$1 = \frac{x}{\theta}, \quad \theta = x$$

MLE is $x=3$, $\theta = 3$.

Example 1: The waiting time between hits for a webpage has an exponential distribution with density f(x)=λe^(-λx). If the first five waiting times are 5.5, 8.0, 2.7, 1.7, and 10.7 seconds, estimate λ.

Joint density is

$$\left(\lambda e^{-\lambda(5.5)} \cdot \lambda e^{-\lambda(8.0)} \cdot \lambda e^{-2.7\lambda}\right)$$

$$\times\left(\lambda e^{-1.7\lambda} \cdot \lambda e^{-10.7\lambda}\right)$$

$$= f(x_1, \ldots, x_5; \lambda) = L(\lambda; \vec{x}) = L(\lambda)$$

$$= \lambda^5 e^{-\lambda[5.5+8+2.7+1.7+10.7]}$$

MLE = λ that max $L(\lambda)$

= λ that max $\ell(\lambda) = \ln L(\lambda)$

$$\ln L(\theta) = \ell(\theta) = 5 \ln \lambda + -\lambda\left(\sum x_i\right)$$

$$0 = \frac{d}{d\theta}\ell(\theta) = \frac{5}{\lambda} - \sum x_i$$

$$\frac{1}{\lambda} = \frac{\sum x_i}{5} = \frac{28.6}{5}, \quad \lambda = .175$$

Example 2: $f(x;\theta) = \theta x^{\theta-1} \quad 0 \le x \le 1$

$x_1 = .2 \quad x_2 = .1 \quad x_3 = .8$

estimate θ

$L(\theta) = \prod_i f(x_i;\theta)$

$= \left[\theta(.2)^{\theta-1}\right]\left[\theta(.1)^{\theta-1}\right]\left[\theta(.8)^{\theta-1}\right]$

$= \theta^3 (.2)^{\theta-1}(.1)^{\theta-1}(.8)^{\theta-1}$

$l(\theta) = 3\ln\theta + (\theta-1)\left[\ln(.2) + \ln(.1) + \ln(.8)\right]$

$0 = l'(\theta) = \frac{3}{\theta} + \left[\ln(.1) + \ln(.2) + \ln(.8)\right]$

$\theta = .725$ is a critical point

Is it a max?

$l''(\theta) = \frac{-3}{\theta^2} < 0$ at $.725$, so

yes, it is a max.

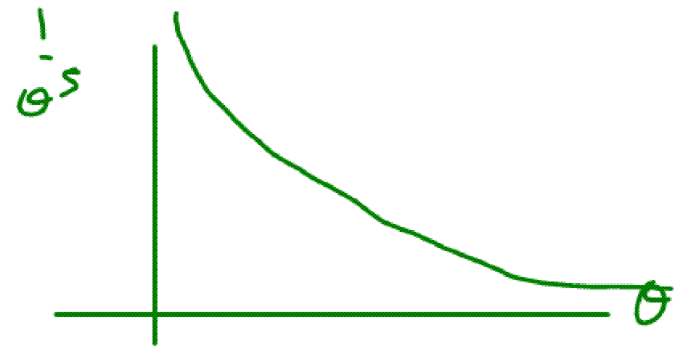
Or: $L(0) = 0^3 \cdot \_ = 0$

$L(.725) > 0 = L(0)$, so it is a max

Example 3: Loss amounts are uniform on $(0,\theta)$. If the first 5 losses are 1.7, 3.0, 1.1, 3.6, and 0.6, find the MLE for θ.

$$f(x;\theta) = \frac{1}{\theta} \quad \text{for } 0 \le x \le \theta$$

$$L(\theta) = \frac{1}{\theta} \cdot \frac{1}{\theta} \cdots \frac{1}{\theta} = \frac{1}{\theta^5}$$



$L(\theta)$ is max when θ is minimized.

Can $\theta = 2$? No, because

2^{nd} data point is $3 > 2$

all data $\le \theta$.

Min θ = max X_i = 3.6

MLE $\theta = 3.6$

If range of X_i depends on θ

MLE is often max/min possible value