Overview

- This chapter goes over some abstract models of defaults, stressing the importance of correlations
- The variable m will be used for the total number of counterparties, and i will be the index for a given counterparty
- Probability space defined by the ordered triplet $(\Omega, F, \mathbb{P})$

Basics and Notation

- Range of possible credit ratings given by $R_i \in \{0, 1, \ldots, d\}$
- d represents the **default state**
- R_i is the credit rating of counterparty i
- $R \to R^{'}$ denotes a rating migration
- If $d = 1$, then this is a simple two-state approach. In this case:
  - $R_i = 0 \to$ No Default
  - $R_i = 1 \to$ Default

Model Summary

- Bernoulli Model (L): Moody's KMV / Risk Metrics
- Poisson Model ($L^{'}$): CreditRisk+

Bernoulli Distribution

- The vector $\mathbf{L} = (L_1, L_2, \cdots, L_m)$ is called a **Bernoulli loss statistic** if all marginal distributions of L_i are given by:

$$L_i \sim B(1; p_i) = \begin{cases} 1 & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases}$$

The total number of defaults is calculated as:

$$\text{Total Defaults} = L = \sum_{i=1}^{m} L_i$$

The *loss percentage* is denoted by:

$$\text{Loss Percentage} = \frac{\text{Total Defaults}}{\text{Total Counterparties}} = \frac{\sum_{i=1}^{m} L_i}{m}$$

Properties

- $p_i = Pr(L_i = 1) = \mathbb{E}(L_i)$ are called the **default probabilities**
- The expected number of defaults is given by $\mathbb{E}(L) = \sum_{i=1}^{m} p_i$
- If independence is assumed, then $\mathbb{V}(L) = \sum_{i=1}^{m} p_i(1 - p_i)$
- If both independence and a uniform default probability p is assumed, then $L \sim \text{Binom}(m; p)$

General Bernoulli Mixture Model

- Correlation is the central challenge in credit porfolio risk. Therefore, the reading goes a step further to build out models with explicit dependencies
- Loss probabilities **P** will be viewed as random variables such that $\mathbf{P} = (P_1, \ldots, P_m) \sim \mathbf{F}$ has some distribution **F**
- Key point: Assume that **conditional on P**, the L's are independent!

General Bernoulli Mixture Model

- Conditional on P, we have that the loss statistics are independent and have binomial distribution given by:

$$L_i|_{P_i=p_i} \sim B(1; p_i) = \begin{cases} 1 & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases}$$

- The first and second moments are given by:
  - $\mathbb{E}(L_i) = \mathbb{E}(P_i)$
  - $\mathbb{V}(L_i) = \mathbb{E}(P_i) \cdot (1 - \mathbb{E}(P_i))$
- $\text{Cov}(L_i, L_j) = \text{Cov}(P_i, P_j)$
- $\text{Corr}(L_1, L_2) = \frac{\text{Cov}(L_1, L_2)}{\sqrt{\mathbb{V}(L_1) \cdot \mathbb{V}(L_2)}} = \frac{\text{Cov}(P_1, P_2)}{\sqrt{\mathbb{E}(P_1) \cdot (1 - \mathbb{E}(P_1)) \cdot \mathbb{E}(P_2) \cdot (1 - \mathbb{E}(P_2))}}$

Uniform Default Probability and Uniform Correlation

- Portfolios with uniform default probability and uniform default correlation are called **uniform portfolios**
- This model works best with portfolios where all exposures are of approximately the same **size** and **type** of risk (since we have the uniform parameter assumptions)
- Two key parameters:
  1. The uniform default probability is denoted by $\bar{p}$
  2. The uniform default correlation is denoted by ρ

Drill Problem #1

Question:

Suppose that for all t,

$$Pr_t(R_i^{t+1} = 1 | R_i^t = 0) = .05$$

What is the probability of default within 6 years assuming $d = 1$?

Solution:

- $1 - (1 - .05)^6 \approx 26.5\%$

- Q: Show how to calculate the expected value of the loss percentage, given $m = 10$ and p_i for $i = 1, 2, \ldots, 10$.

Solution:

$$E(\text{Loss Percentage}) = \frac{\mathrm{E}\left(\sum_{i=1}^{10} L_i\right)}{10} = \frac{\sum_{i=1}^{10} E(L_i)}{10} = \boxed{\frac{\sum_{i=1}^{10} p_i}{10}}$$

Drill Problem #3

- Q: Consider the following two-dimensional Generalized Bernoulli Mixture Model:
- $F = (P_1, P_2)$ where $P_2 = 1.17 \cdot P_1$ and $P_1 \sim U[0, .1]$
- Given the information above, complete the following:
  (a) Describe the values that L_1 and L_2 can take
  (b) Describe the values that P_1 and P_2 can take
  (c) Determine $\mathbb{E}(L_1 + L_2)$
  (d) Determine $\text{Cov}(L_1, L_2)$
  (e) Determine $\mathbb{V}(L_1 + L_2)$
  (f) Determine $\text{Corr}(L_1, L_2)$
  (g) Which firm is riskier?
  (h) Calculate $\mathbb{E}(L_1 L_2)$

Hint 1: If $U \sim [0, b]$, *then* $\mathbb{V}(U) = \frac{1}{12}b^2$
Hint 2: If $U \sim [0, b]$, *then* $\mathbb{E}(U^2) = \frac{1}{3}b^2$

(a) L_1 can either be 0 or 1. Similarly, L_2 can either be 0 or 1. A value of 1 corresponds to default. A value of 0 corresponds to no default.

(b) P_1 can take any value from 0 to .1. Similarly, P_2 can take any value from 0 to .117.

(c) $\mathbb{E}(L_1 + L_2) = \mathbb{E}(L_1) + \mathbb{E}(L_2) = \mathbb{E}(P_1) + \mathbb{E}(P_2) =$ $.05 + 1.17 \cdot .05 \approx \boxed{.1085}$

(d) $\text{Cov}(L_1, L_2) = \text{Cov}(P_1, P_2) = \text{Cov}(P_1, 1.17 \cdot P_1) =$ $1.17 \cdot \text{Cov}(P_1, P_1)$
$= 1.17 \cdot \mathbb{V}(P_1) = 1.17 \cdot \frac{1}{12} \cdot .1^2 \approx \boxed{.000975}$

(e) $\mathbb{V}(L_1 + L_2) = \mathbb{V}(L_1) + \mathbb{V}(L_2) + 2 \cdot \text{Cov}(L_1, L_2)$
$.05 \cdot (1 - .05) + 1.17 \cdot .05 \cdot (1 - 1.17 \cdot .05) + 2 \cdot .000975 \approx \boxed{.104528}$

(f) $\text{Corr}(L_1, L_2) = \frac{\text{Cov}(L_1, L_2)}{\sqrt{\mathbb{V}(L_1)\cdot\mathbb{V}(L_2)}}$

$= \frac{.000975}{\sqrt{(.05)(1-.05)\cdot(1.17)(.05)(1-1.17\cdot.05)}} = \boxed{.01906205}$

(g) Firm 2 is riskier, because it has the 1.17 multiplier.

(h) Note that for (h), you could use either approach below. For Method II, apply Hint 2.

Method I: $\text{Cov}(L_1, L_2) = \mathbb{E}(L_1 L_2) - \mathbb{E}(L_1)\mathbb{E}(L_2)$

$\Rightarrow .000975 = \mathbb{E}(L_1 L_2) - .05 \cdot 1.17(.05) \Rightarrow \mathbb{E}(L_1 L_2) = \boxed{.0039}$

Method II:

$\mathbb{E}(L_1 L_2) = \mathbb{E}(P_1 P_2) = 1.17 \cdot \mathbb{E}(P_1^2) = 1.17 \cdot \frac{.1^2}{3} \approx \boxed{.0039}$