A.1 Probability Review - Outline

A.1.2 Moments

E$[X]$ and E$[g(X)]$

CV, Skewness and Kurtosis

Exercises

Moments

If X is discrete:

$$\mathrm{E}[X] = \sum_x x \cdot \mathrm{P}[X = x]$$

$$\mathrm{E}\left[X^k\right] = \sum_x x^k \cdot \mathrm{P}[X = x]$$

$$\mathrm{E}[g(X)] = \sum_x g(x) \cdot \mathrm{P}[X = x]$$

If X is continuous

$$\mathrm{E}[X] = \int x \cdot f(x)dx$$

$$\mathrm{E}\left[X^k\right] = \int x^k \cdot f(x)dx$$

$$\mathrm{E}[g(X)] = \int g(x) \cdot f(x)dx$$

For a mixed distribution:

- Use discrete formula over discrete values
- Use continuous formula over continuous values
- Sum the two pieces

Survival Function Approach

Recall that $S(x) = \mathrm{P}[X > x]$.
For any loss variable X, integration by parts gives

$$\begin{aligned}\mathrm{E}[X] &= \int_0^\infty x \cdot f(x)\,dx \\ &= x \cdot [-S(x)]\Big|_0^\infty + \int_0^\infty S(x)dx \\ &= 0 + \int_0^\infty S(x)dx\end{aligned}$$

Also true, but less useful: If $g(0) = 0$,

$$\mathrm{E}[g(X)] = \int_0^\infty g'(x) \cdot S(x)dx$$

Central Moments

$$\mathrm{E}\left[X^k\right] = \mu'_k = k\text{-th } \textit{raw} \text{ moment.}$$

$$\mu'_1 = \mu = \mathrm{E}[X]$$

$$\mathrm{E}\left[(X-\mu)^k\right] = \mu_k = k\text{-th } \textit{central} \text{ moment}$$

$$\mu_2 = \sigma^2 = \mathrm{Var}[X]$$

$$\mathrm{Var}[X] = \mathrm{E}\left[(X-\mu)^2\right] = \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2$$

Bernoulli shortcut: If $\mathrm{P}[X = a] + \mathrm{P}[X = b] = 1$ then

$$\mathrm{Var}[X] = (b-a)^2 \cdot \mathrm{P}[X = a] \cdot \mathrm{P}[X = b]$$

If $Y = aX$ then $\mathrm{E}[Y^k] = \mathrm{E}[(aX)^k] = a^k \mathrm{E}[X^k]$

$$\mu'_k(Y) = a^k\,\mu'_k(X)$$

$$\mu_k(Y) = a^k\,\mu_k(X)$$

Converting Raw to Central Moments

$$
\begin{aligned}
\mathrm{E}\big[(X-\mu)^2\big] &= \mathrm{Var}[X] = \sigma^2 \\
&= \mathrm{E}\big[X^2 - 2\mu X + \mu^2\big] = \mathrm{E}\big[X^2\big] - 2\mu \mathrm{E}[X] + \mu^2 \\
&= \mathrm{E}\big[X^2\big] - 2\mu \cdot \mu + \mu^2 = \mathrm{E}\big[X^2\big] - \mu^2 \\
\mathrm{E}\big[(X-\mu)^3\big] &= \mathrm{E}\big[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3\big] \\
&= \mathrm{E}\big[X^3\big] - 3\mu \mathrm{E}\big[X^2\big] + 3\mu^2 \cdot \mathrm{E}[X] - \mu^3 \\
&= \mathrm{E}\big[X^3\big] - 3\mu \mathrm{E}\big[X^2\big] + 3\mu^2 \cdot \mu - \mu^3 \\
&= \mathrm{E}\big[X^3\big] - 3\mu \mathrm{E}\big[X^2\big] + 2\mu^3
\end{aligned}
$$

You can do a similar thing with any central moment. This is rarely tested – remember the process, don't memorize the results!

Coefficient of Variation

$\mu = \mathrm{E}[X] \quad \sigma^2 = \mathrm{Var}[X] \quad \sigma = \mathrm{SD}[X]$

Definition (Coefficient of Variation)

$\mathrm{CV}[X] = \dfrac{\sigma}{\mu}$

Note that for $c > 0$,

$$
\begin{aligned}
\mathrm{CV}[X] &= \frac{\sigma}{\mu} = \frac{\mathrm{SD}[X]}{\mathrm{E}[X]} \\
\mathrm{CV}[cX] &= \frac{\mathrm{SD}[cX]}{\mathrm{E}[cX]} = \frac{c\mathrm{SD}[X]}{c\mathrm{E}[X]} \\
&= \mathrm{CV}[X]
\end{aligned}
$$

Skewness and Kurtosis

Definition (Skewness)

$$\text{Skewness}(X) = \frac{\mathrm{E}\left[(X-\mu)^3\right]}{\sigma^3}$$

Definition (Kurtosis)

$$\text{Kurtosis}(X) = \frac{\mathrm{E}\left[(X-\mu)^4\right]}{\sigma^4}$$

The powers in numerator and denominator match, so for $c > 0$,
Skewness(cX) = Skewness(X) and Kurtosis(cX)=Kurtosis(X).

X is symmetric $\Rightarrow$ odd central moments are 0.
Skewness(X)=0 for any symmetric (and thus non-skewed) distribution.

Exercise 1

$\mathrm{P}[X = 100] = 0.3$ and $\mathrm{P}[X = 300] = 0.7$. Find the skewness of X.

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P$[X = 100] = 0.3$ and P$[X = 300] = 0.7$. Find the skewness of X.

$$\begin{aligned}
\mathrm{E}[X] &= 0.3 \cdot 100 + 0.7 \cdot 300 = 240 \\
\mathrm{Var}[X] &= 0.3 \cdot (100 - 240)^2 + 0.7 \cdot (300 - 240)^2 \\
&= (300 - 100)^2 \cdot 0.3 \cdot 0.7 = 8{,}400 \\
\mathrm{E}[(X - \mu)^3] &= 0.3 \cdot (100 - 240)^3 + 0.7 \cdot (300 - 240)^3 \\
&= -672{,}000 \\
\mathrm{Skew}[X] &= \frac{\mathrm{E}[(X - \mu)^3]}{\sigma^3} \\
&= \frac{-672{,}000}{8{,}400^{3/2}} \\
&= \boxed{-0.873}
\end{aligned}$$

Exercise 2

X has hazard rate $h(x) = 3/x$ for $x > 4$ and 0 for $x \leq 4$. Find E$[X]$.

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X has hazard rate $h(x) = 3/x$ for $x > 4$ and 0 for $x \le 4$. Find $\mathrm{E}[X]$.

$$\begin{aligned} H(x) &= 0 \quad x \le 4 \\ &= \int_{-\infty}^{x} h(t)\,dt = H(4) + \int_{4}^{x} h(t)\,dt \quad x > 4 \\ &= 0 + 3\ln(t)\Big|_4^x = \ln[(x/4)^3] \quad x > 4 \\ S(x) &= \begin{cases} 1 & x \le 4 \\ \dfrac{4^3}{x^3} & x > 4 \end{cases} \\ \mathrm{E}[X] &= \int_0^\infty S(x)dx = \int_0^4 1\,dx + \int_4^\infty \frac{4^3}{x^3}dx \\ &= 4 + \frac{-1}{2}\cdot\frac{4^3}{x^2}\Big|_4^\infty = 4 + \frac{4}{2} = \boxed{6} \end{aligned}$$

Or: X is a single parameter Pareto with $\alpha = 3$ and $\theta = 4$, so

$\mathrm{E}[X] = \dfrac{\alpha\theta}{\alpha - 1} = \dfrac{3\cdot 4}{3-1} = \boxed{6}$