The Infinite Actuary Exam STAM Online Course
A.1.2. Moments
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1. A nonnegative random variable X has a hazard rate function of $h(x) = \dfrac{3}{x}$ for $x > 2$ and 0 otherwise. Find $\mathrm{E}[X]$.

   A. 1 B. 2 C. 3 D. 6 E. 9

$$H(x) = \int_0^2 0\,dt + \int_2^x \frac{3}{t}dt = 3\ln\left(\frac{x}{2}\right) \quad x > 2$$

$$S(x) = e^{-H(x)} = \begin{cases} 1 & x \le 2 \\ \dfrac{8}{x^3} & x > 2 \end{cases}$$

$$\mathrm{E}[X] = \int_0^2 1\,dx + \int_2^\infty \frac{8}{x^3}dx = 2 + \frac{4}{2^2} = \boxed{3}$$

$$\text{or: } f(x) = -S'(x) = \frac{24}{x^4} \quad x > 2$$

$$\mathrm{E}[X] = \int_2^\infty x \cdot \frac{24}{x^4}dx = \int_2^\infty \frac{24}{x^3}dx = \frac{12}{2^2} = \boxed{3}$$

Or: $X \sim$ single parameter Pareto$(\alpha = 3, \theta = 2)$ and $\mathrm{E}[X] = \dfrac{\alpha\theta}{\alpha - 1} = \dfrac{3 \cdot 2}{2} = 3$

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2. The survival function of X is $2 - x$ for $1 < x < 2$. Find $\mathrm{E}[X]$.

   A. 1/2 B. 5/4 C. 4/3 D. 3/2 E. 2

$$\mathrm{E}[X] = \int_0^\infty S(x)\,dx = \int_0^1 1\,dx + \int_1^2 (2 - x)dx$$

$$= 1 + 2 - \frac{2^2 - 1^2}{2} = \boxed{1.5}$$

$$\text{or: } f(x) = -S'(x) = 1 \quad 1 < x < 2$$

$$\mathrm{E}[X] = \int_1^2 x \cdot 1\,dx = \frac{2^2 - 1^2}{2} = \boxed{1.5}$$

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3. The survival function of X is $\dfrac{4 - x^2}{3}$ for $1 < x < 2$. Find $\mathrm{Var}[X]$.

   A. 0.080 B. 0.081 C. 0.082 D. 0.083 E. 0.084

$$f(x) = -S'(x) = \frac{2x}{3} \quad 1 < x < 2$$

$$\mathrm{E}[X] = \int_1^2 x \cdot \frac{2x}{3}\,dx = \frac{2}{9}\left(2^3 - 1^3\right) = \frac{14}{9}$$

$$\mathrm{E}[X^2] = \int_1^2 x^2 \cdot \frac{2x}{3}\,dx = \frac{2}{12}\left(2^4 - 1^4\right) = \frac{5}{2}$$

$$\text{Var}[X] = \frac{5}{2} - \left(\frac{14}{9}\right)^2 = \frac{13}{162} = \boxed{0.080}$$

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4. [4.S01.3] You are given the following times of first claim for five randomly selected auto insurance policies observed from time $t = 0$:

$$1 \quad 2 \quad 3 \quad 4 \quad 5$$

Calculate the kurtosis of this sample.

A. 0.0 B. 0.5 C. 1.7 D. 3.4 E. 6.8

$$\sigma^2 = \frac{(-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2}{5} = 2 \text{ and } \mu_4 = \frac{(-2)^4 + (-1)^4 + 0^4 + 1^4 + 2^4}{5} = \frac{34}{5}$$

so the kurtosis is $(34/5)/4 = \boxed{1.7}$

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5. [3-CAS.F04.28] A large retailer of computers issues a warranty with each computer it sells. The warranty covers any cost to repair or replace a defective computer within 30 days of purchase. 40% of all claims are easily resolved and do not involve any cost to replace or repair. If a claim involves a cost to replace or repair, the claim size is distributed as a Weibull with parameters $\tau = \frac{1}{2}$ and $\theta = 30$.

Which of the following statements are true?

(i) The expected cost of a claim is \$60.

(ii) The survival function at \$60 is 0.243.

(iii) The hazard rate at \$60 is 0.012.

A. (i) only.
B. (ii) only.
C. (iii) only.
D. (i) and (ii) only.
E. (ii) and (iii) only.

From the exam tables, the mean of a Weibull distribution is $\theta\Gamma\left(1 + \frac{1}{\tau}\right)$. To evaluate that, we will use the fact that $\Gamma(x) = (x-1)!$ when x is an integer. Since only 60% of claims result in a payment, we have $\text{E}[\text{cost}] = 0.6 \cdot \text{E}[\text{Weibull}] = 0.6 \cdot 30(3-1)! = 36$ so (i) is false and we can eliminate A and D.

The survival function at 60 is 0.6 times the survival function of a Weibull at 60, giving us $0.6\left[e^{-(60/30)^{1/2}}\right] = 0.6 \cdot 0.243 = 0.146$ so (ii) is false and we can eliminate B and E.

We now know the answer is $\boxed{C}$ (only choice left!) but let's confirm that (iii) holds:

$$h(60) = \frac{f(60)}{S(60)} = \frac{0.6\left[\frac{1}{2}\left(\frac{60}{30}\right)^{1/2}\frac{1}{60}e^{-\sqrt{2}}\right]}{0.6e^{-\sqrt{2}}} = 0.01179.$$

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6. [C.F06.3] You are given a random sample of 10 claims consisting of two claims of 400, seven claims of 800, and one claim of 1600. Determine the empirical skewness coefficient.

A. Less than 1.0
B. At least 1.0, but less than 1.5
C. At least 1.5, but less than 2.0
D. At least 2.0, but less than 2.5
E. At least 2.5

$$\begin{aligned}\mu &= \frac{1}{10}\left[2 \cdot 400 + 7 \cdot 800 + 1600\right] = 800 \\ \sigma^2 &= (0.2)(-400)^2 + (0.1)(800)^2 = 96{,}000 \\ \mu_3 &= (0.2)(-400)^3 + (0.1)(800)^3 = 38{,}400{,}000\end{aligned}$$

The skewness is thus $38{,}400{,}000/(96{,}000)^{3/2} = \boxed{1.29}$

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7. [4B.S95.28] You are given the following:

- For any random variable X with finite first three moments, the skewness of the distribution of X is denoted $\mathrm{Sk}(X)$.
- X and Y are independent, identically distributed random variables with mean 0 and finite second and third moments.

Which of the following statements must be true?

(i) $2\mathrm{Sk}(X) = \mathrm{Sk}(2X)$
(ii) $-\mathrm{Sk}(Y) = \mathrm{Sk}(-Y)$
(iii) $|\mathrm{Sk}(X)| \geq |\mathrm{Sk}(X+Y)|$

A. (ii) only B. (iii) only C. (i) and (ii) only D. (ii) and (iii) only E. None of A, B, C, or D

$\mathrm{Sk}(cX) = \mathrm{Sk}(X)$ for $c > 0$, so (i) is false.
$\mathrm{SD}[-Y] = \mathrm{SD}[Y]$ but $\mathrm{E}\left[\left((-Y) - \mathrm{E}[-Y]\right)^3\right] = -\mathrm{E}\left[(Y - \mathrm{E}[Y])^3\right]$ so $\mathrm{Sk}(-Y) = -\mathrm{Sk}(Y)$ and (ii) is true.
Since X and Y are iid with mean 0,

$$\begin{aligned}\mathrm{Sk}(X+Y) &= \frac{\mathrm{E}\left[(X+Y-0-0)^3\right]}{(\mathrm{SD}[X+Y])^3} \\ &= \frac{\mathrm{E}[X^3] + 3\mathrm{E}[X^2Y] + 3\mathrm{E}[XY^2] + \mathrm{E}[Y^3]}{(\mathrm{Var}[X+Y])^{3/2}} \\ &= \frac{\mathrm{E}[X^3] + 3\mathrm{E}[X^2]\cdot 0 + 3\cdot 0\cdot \mathrm{E}[Y^2] + \mathrm{E}[Y^3]}{2^{3/2}(\mathrm{Var}[X])^{3/2}} \\ &= \frac{2\mathrm{E}[X^3]}{2^{3/2}\mathrm{SD}[X])^3} = \frac{\mathrm{Sk}(X)}{\sqrt{2}}\end{aligned}$$

so (iii) is true and the answer is $\boxed{D}$

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8. You observe the following losses:

Loss amount	0	100	200	300	400	500	600
Number of losses	12	38	26	12	9	1	2

Calculate the empirical coefficient of variation for the loss data.

A. 0.01 B. 0.73 C. 1.37 D. 10.58 E. 50.79

To do this on the calculator, you can put the loss amounts in L1 in the data mode, and number of losses in L2. Then 1-Var stats with data = L1 and FRQ = L2 gives $\mu = 179$ and $\sigma = 130.6$. To do it by hand, note that we have 100 total data points, so

$$\begin{aligned}\mu &= 0 \cdot \frac{12}{100} + 100 \cdot \frac{38}{100} + 200 \cdot \frac{26}{100} + 300 \cdot \frac{12}{100} + 400 \cdot \frac{9}{100} + 500 \cdot \frac{1}{100} + 600 \cdot \frac{2}{100} \\ &= 179 \\ \mu_2' &= 0^2 \cdot \frac{12}{100} + 100^2 \cdot \frac{38}{100} + 200^2 \cdot \frac{26}{100} + 300^2 \cdot \frac{12}{100} + 400^2 \cdot \frac{9}{100} + 500^2 \cdot \frac{1}{100} + 600^2 \cdot \frac{2}{100} \\ &= 49{,}100\end{aligned}$$

so the empirical standard deviation is $\sqrt{49{,}100 - 179^2} = 130.6$ and the CV is $130.6/179 = \boxed{0.73}$

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9. [3-CAS.S04.28] A pizza delivery company has purchased an automobile liability policy for its delivery drivers from the same insurance company for the past five years. The number of claims filed by the pizza delivery company as the result of at-fault accidents caused by its drivers is shown below:

Year	Claims
2002	4
2001	1
2000	3
1999	2
1998	15

Calculate the skewness of the empirical distribution of the number of claims per year.

A. Less than 0.50
B. At least 0.50, but less than 0.75
C. At least 0.75, but less than 1.00
D. At least 1.00, but less than 1.25
E. At least 1.25

$\mu = \dfrac{4+1+3+2+15}{5} = 5$, and the skewness is $\dfrac{\mathrm{E}[(X-\mu)^3]}{\sigma^3} = \dfrac{\frac{1}{5} \cdot 900}{\left(\frac{1}{5}130\right)^{3/2}} = \boxed{1.36}$

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10. [4B.S93.34] Claim severity has the following distribution:

Claim Size	100	200	300	400	500
Probability	0.05	0.20	0.50	0.20	0.05

Determine the distribution's skewness coefficient.

A. -0.25 B. 0 C. 0.15 D. 0.35 E. Cannot be determined

The distribution is symmetric about 300, so the mean is 300 and the skewness is 0. Alternatively,

$$\begin{aligned}
\mathrm{E}[X] &= 0.05 \cdot 100 + 0.2 \cdot 200 + \cdots + 0.05 \cdot 500 = 300 \\
\mathrm{E}[(X-300)^3] &= 0.05 \cdot (-200)^3 + 0.2 \cdot (-100)^3 + \cdots + 0.05 \cdot 200^3 = 0 \\
\mathrm{Skew}[X] &= \frac{0}{(\mathrm{SD}[X])^3} = \boxed{0}
\end{aligned}$$

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11. The following claims are observed:

Claim Size	Number of Claims
2,000	2
4,000	6
6,000	12
8,000	10

What is the empirical skewness coefficient?

A. -1.00 B. -0.56 C. 1.49 D. 2.50 E. 2.67

Since skewness is scale invariant, we can divide all of our claim sizes by 1,000 to simplify the numbers. Let X denote the claim size, and $Y = X/1{,}000$. Then

$$\begin{aligned}
\mathrm{E}[Y] &= \frac{2 \cdot 2 + 4 \cdot 6 + 6 \cdot 12 + 8 \cdot 10}{2 + 6 + 12 + 10} = 6 \\
\mathrm{E}[Y^2] &= \frac{2^2 \cdot 2 + 4^2 \cdot 6 + 6^2 \cdot 12 + 8^2 \cdot 10}{30} = 39.2 \\
\mathrm{SD}[Y] &= \sqrt{39.2 - 6^2} = \sqrt{3.2} \\
\mathrm{E}[(Y-\mu_Y)^3] &= \frac{(2-6)^3 \cdot 2 + (4-6)^3 \cdot 6 + (6-6)^3 \cdot 12 + (8-6)^3 \cdot 10}{30} = -3.2 \\
\mathrm{Skew}[Y] &= \frac{-3.2}{(\sqrt{3.2})^3} = \boxed{-0.56}
\end{aligned}$$

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12. Suppose that X is exponential with mean 10, and Y is an independent exponential with mean 20. Find $\text{Var}[XY]$.

A. 20,000 B. 60,000 C. 80,000 D. 120,000 E. 160,000

$$\begin{aligned} \text{E}[XY] &= \text{E}[X] \cdot \text{E}[Y] = 10 \cdot 20 = 200 \\ \text{E}\big[(XY)^2\big] &= \text{E}\big[X^2\big] \cdot \text{E}\big[Y^2\big] \\ &= 2 \cdot 10^2 \cdot 2 \cdot 20^2 = 160{,}000 \\ \text{Var}[XY] &= 160{,}000 - 200^2 = \boxed{120{,}000} \end{aligned}$$