The Infinite Actuary Exam STAM Online Course
A.1.3. Generating Functions
Last updated April 11, 2018

1. A discrete distribution N has probability generating function

$$P_N(z) = (0.3 + 0.2z + 0.5z^2)^5$$

Find $\mathrm{P}[N = 2]$.

A. 0.023 B. 0.031 C. 0.046 D. 0.055 E. 0.062

$$\begin{aligned}
\mathrm{P}[N=2] &= \frac{P''(0)}{2} \\
P'(z) &= 5\left(0.3 + 0.2z + 0.5z^2\right)^4 \cdot (0.2 + z) \\
P''(z) &= 20\left(0.3 + 0.2z + 0.5z^2\right)^3 \cdot (0.2 + z)^2 + 5\left(0.3 + 0.2z + 0.5z^2\right)^4 \\
P''(0) &= 20 \cdot 0.3^3 \cdot 0.2^2 + 5 \cdot 0.3^4 = 0.0621 \\
\mathrm{P}[N=2] &= \frac{0.0621}{2} = \boxed{0.031}
\end{aligned}$$

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2. A discrete distribution N has probability generating function

$$P_N(z) = (0.3 + 0.2z + 0.5z^2)^5$$

Find $\mathrm{Var}[N]$.

A. 1.9 B. 3.8 C. 7.6 D. 33.8 E. 39.8

See previous problem for $P'(z)$ and $P''(z)$.

$$\begin{aligned}
\mathrm{E}[N] &= P'(1) = 5 \cdot 1^4 \cdot 1.2 = 6 \\
\mathrm{E}[N(N-1)] &= P''(1) = 20 \cdot 1^3 \cdot 1.2^2 + 5 \cdot 1^4 = 33.8 \\
\mathrm{E}[N^2] - \mathrm{E}[N] &= \mathrm{E}[N^2] - 6 = 33.8 \Rightarrow \mathrm{E}[N^2] = 39.8 \\
\mathrm{Var}[N] &= 39.8 - 6^2 = \boxed{3.8}
\end{aligned}$$

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3. An actuary models the number of losses using a distribution with probability generating function

$$P(z) = 1 - (1 - z)^{1/4}, \quad z < 1$$

According to the model, what is the probability of having exactly 3 losses?

A. $\frac{3}{64}$ B. $\frac{7}{128}$ C. $\frac{3}{16}$ D. $\frac{7}{32}$ E. $\frac{21}{64}$

$$\mathrm{P}[N=3] = \frac{P'''(0)}{3!}$$

$$
\begin{aligned}
P'(z) &= \frac{1}{4}(1-z)^{-3/4} \\
P''(z) &= \frac{1}{4}\cdot\frac{3}{4}\cdot(1-z)^{-7/4} \\
P'''(z) &= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{7}{4}\cdot(1-z)^{-11/4} \\
P'''(0) &= \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{7}{4} = \frac{21}{64} \\
\mathrm{P}[N=3] &= \frac{21}{64}\cdot\frac{1}{6} = \boxed{\frac{7}{128}}
\end{aligned}
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4. [3.F06.25] You are given the following information about the probability generating function for a discrete distribution:

$$P'(1) = 2 \qquad P''(1) = 6$$

Calculate the variance of the distribution.

A. Less than 1.5
B. At least 1.5, but less than 2.5
C. At least 2.5, but less than 3.5
D. At least 3.5, but less than 4.5
E. At least 4.5

$P'(1) = 2 = \mathrm{E}[X]$, and $P''(1) = \mathrm{E}[X(X-1)] = 6$. Expanding the second equation gives us $\mathrm{E}[X^2 - X] = \mathrm{E}[X^2] - \mathrm{E}[X] = \mathrm{E}[X^2] - 2 = 6$, so $\mathrm{E}[X^2] = 8$ and $\mathrm{Var}(X) = 8 - 2^2 = \boxed{4}$

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5. The moment generating function of X is $M_X(t) = e^{2t^2-5t}$. Find $\mathrm{Var}[X]$.

A. 1 B. 2 C. 3 D. 4 E. 5

$M'(t) = (4t-5)e^{2t^2-5t}$ so $\mathrm{E}X = M'(0) = -5e^0 = -5$.
$M''(t) = 4e^{2t^2-5t} + (4t-5)^2 e^{2t^2-5t}$ so $\mathrm{E}[X^2] = 4e^0 + (-5)^2 e^0 = 4 + 5^2$.
This gives $\mathrm{Var}(X) = (4+5^2) - (-5)^2 = \boxed{4}$

Remark: $M_X(t)$ is the MGF of a normal random variable with mean -5 and variance $2\cdot 2 = 4$, but you don't need to know that for the exam.

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6. You are given that the probability generating function of a random variable X is

$$P_X(z) = \frac{1}{4-3z}$$

Find the second raw moment of X.

A. 3 B. 9 C. 12 D. 18 E. 21

$P'(z) = 3(4-3z)^{-2}$ so $\mathrm{E}[X] = P'(1) = 3$.
$P''(z) = 18(4-3z)^{-3}$ so $\mathrm{E}[X(X-1)] = P''(1) = 18$.
Expanding, we get $\mathrm{E}[X(X-1)] = \mathrm{E}[X^2] - \mathrm{E}[X]$ so $18 = \mathrm{E}[X^2] - 3$ and $\mathrm{E}[X^2] = \boxed{21}$

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7. An actuary models the number of losses using a Sibuya distribution with probability generating function

$$P(z) = 1-(1-z)^{1/3}, \quad z<1$$

According to the model, what is the probability of having 2 losses?

A. 1/18 B. 1/9 C. 1/6 D. 2/9 E. 1/3

Since we are talking about the number of losses, our random variable N is discrete. That means that

$$P_N(z) = \mathrm{E}\left[z^N\right] = 1\cdot \mathrm{P}[N=0] + z\cdot \mathrm{P}[N=1] + z^2\cdot \mathrm{P}[N=2] + \dots$$

In particular, $P''(0) = 2\mathrm{P}[N=2]$, which here means that

$$\begin{aligned}
P(z) &= 1-(1-z)^{1/3} \\
P'(z) &= \frac{1}{3}(1-z)^{-2/3} \\
P''(z) &= \frac{1}{3}\cdot\frac{2}{3}(1-z)^{-5/3} \\
P''(0) &= \frac{2}{9} \\
\mathrm{P}[N=2] &= \frac{1}{2}P''(0) = \frac{1}{2}\cdot\frac{2}{9} = \boxed{\frac{1}{9}}
\end{aligned}$$

Or, using Newton's generalization of the binomial theorem

$$(a+b)^x = a^x + \frac{x}{1}a^{x-1}\cdot b + \frac{x(x-1)}{2!}a^{x-2}\cdot b^2 + \dots$$

we can expand $P(z)$ to obtain

$$\begin{aligned}
P(z) &= 1-(1-z)^{1/3} \\
&= 1-\left[1^{1/3} + \frac{1/3}{1}1^{1/3-1}\cdot(-z) + \frac{(1/3)(-2/3)}{2}1^{1/3-2}\cdot(-z)^2 + \dots\right] \\
&= \frac{z}{3} + \frac{z^2}{9} + \dots
\end{aligned}$$

so $\mathrm{P}[N = 2] = \boxed{\dfrac{1}{9}}$
