A.1 Probability Review - Outline

Moment Generating Functions (MGFs)

Rarely tested on STAM, more of an Exam P topic.
Mostly comes up with frailty distributions and calculus tricks.

Definition (MGF)

$M_X(t) = \mathrm{E}\left[e^{tX}\right] = \mathrm{E}\left[\left(e^t\right)^X\right]$

$$
\begin{aligned}
M_X(t) &= \mathrm{E}\left[e^{tX}\right] & M_X(0) &= \mathrm{E}\left[e^{0\cdot X}\right] = 1 \\
\frac{d}{dt}M_X(t) = M_X'(t) &= \mathrm{E}\left[Xe^{tX}\right] & M_X'(0) &= \mathrm{E}[X] \\
\frac{d^2}{dt^2}M_X(t) = M_X''(t) &= \mathrm{E}\left[X^2e^{tX}\right] & M_X''(0) &= \mathrm{E}\left[X^2\right] \\
&\textit{etc.}
\end{aligned}
$$

Cumulant Generating Function

Even less likely than MGFs to be useful on STAM.
Cumulant generating function give the 2nd and 3rd central moments.

$$
\begin{aligned}
g(t) &= \ln[M_X(t)] = \ln\left(\mathrm{E}\left[e^{tX}\right]\right) \\
g(0) &= \ln[M_X(0)] = \ln(1) = 0 \\
g'(0) &= \mu = \mathrm{E}[X] \\
g''(0) &= \sigma^2 = \mathrm{Var}[X] \\
g'''(0) &= \mathrm{E}[(X-\mu)^3] \\
g''''(0) &= \text{messy}
\end{aligned}
$$

This is rarely / never useful on STAM. If we know the MGF, we typically know the distribution and therefore $\mathrm{Var}(X)$. For the 3rd central moment, we will rarely know the MGF so we will rarely (never?) use the cumulant generating function.

Probability Generating Functions

Tested more often (but still not every exam)

Definition (Probability Generating Function)

$$P_X(z) = \mathrm{E}\left[z^X\right] = \mathrm{E}\left[\left(e^{\ln z}\right)^X\right] = M_X(\ln z)$$

If X is discrete, then

$$
\begin{aligned}
P_X(z) &= 1 \cdot \mathrm{P}[X=0] + z \cdot \mathrm{P}[X=1] + z^2 \cdot \mathrm{P}[X=2] + \ldots \\
P(0) &= 1 \cdot \mathrm{P}[X=0] \\
P'(z) &= \qquad 0 \qquad + \quad \mathrm{P}[X=1] + 2z \cdot \mathrm{P}[X=2] + \ldots \\
P'(0) &= \mathrm{P}[X=1] \\
P''(0) &= 2\mathrm{P}[X=2], \quad \mathrm{P}[X=2] = \frac{P''(0)}{2}
\end{aligned}
$$

$$\mathrm{P}[X=k] = P_X^{(k)}(0)/k!$$

Exam tables include the MGF for some continuous distributions, and the PGF for discrete distributions.

PGFs and Moments

$M^{(k)}(0) = \mathrm{E}[X^k]$.

We can also find moments by taking derivatives of the PGF at 1.

$$
\begin{aligned}
P_X(z) &= \mathrm{E}\big[z^X\big] & P_X(1) &= \mathrm{E}\big[1^X\big] = 1 \\
P'_X(z) &= \mathrm{E}\big[Xz^{X-1}\big] & P'_X(1) &= \mathrm{E}[X] \\
P''_X(z) &= \mathrm{E}\big[X(X-1)z^{X-2}\big] & P''_X(1) &= \mathrm{E}[X(X-1)]
\end{aligned}
$$

The derivatives at 1 are called “factorial” moments of X.

Example

Let N be a discrete random variable with

$$p_1 = 0.2 \quad p_2 = 0.4 \quad p_3 = 0.3 \quad p_4 = 0.1$$

What is the probability generating function of N? What is the second moment?

$$
\begin{aligned}
P_N(z) &= \mathrm{E}[z^N] = 0.2 \cdot z^1 + 0.4 \cdot z^2 + 0.3 \cdot z^3 + 0.1 \cdot z^4 \\
\mathrm{E}[N^2] &= 0.2 \cdot 1^2 + 0.4 \cdot 2^2 + 0.3 \cdot 3^2 + 0.1 \cdot 4^2 = \boxed{6.1} \\
\text{Or: } P'(z) &= 0.2 + 0.8z + 0.9z^2 + 0.4z^3 \\
P''(z) &= \ 0 \ + 0.8 + 1.8z + 1.2z^2 \\
P'(1) &= 0.2 + 0.8 + 0.9 + 0.4 = 2.3 \\
P''(1) &= 0.8 + 1.8 + 1.2 = 3.8 \\
\mathrm{E}[N(N-1)] &= \mathrm{E}[N^2] - \mathrm{E}[N] \\
3.8 &= \mathrm{E}[N^2] - 2.3 \\
\mathrm{E}[N^2] &= \boxed{6.1}
\end{aligned}
$$

Exercise 1

The probability generating function for a discrete variable N satisfies the following:

$$P'(0) = \frac{3}{16} \quad P'(1) = 3 \quad P''(0) = \frac{9}{32} \quad P''(1) = 18$$

Find $\mathrm{P}[N = 2]$.

Exercise 1

The probability generating function for a discrete variable N satisfies the following:

$$P'(0) = \frac{3}{16} \quad P'(1) = 3 \quad P''(0) = \frac{9}{32} \quad P''(1) = 18$$

Find $\mathrm{P}[N = 2]$.

$$\begin{aligned}
P_N(z) &= \mathrm{P}[N=0] + z \cdot \mathrm{P}[N=1] + z^2 \cdot \mathrm{P}[N=2] + z^3 \cdot \mathrm{P}[N=3] + \ldots \\
P_N'(z) &= \quad 0 \quad + \mathrm{P}[N=1] + 2z \cdot \mathrm{P}[N=2] + 3z^2 \cdot \mathrm{P}[N=3] + \ldots \\
P_N''(z) &= \quad 0 \quad + 2 \cdot \mathrm{P}[N=2] + 6z \cdot \mathrm{P}[N=3] + \ldots \\
P_N''(0) &= 2 \cdot \mathrm{P}[N=2] \\
\frac{9}{32} &= 2 \cdot \mathrm{P}[N=2]
\end{aligned}$$

$$\mathrm{P}[N=2] = \boxed{\frac{9}{64}}$$

Exercise 2

The probability generating function for a discrete distribution satisfies the following:

$$P'(0) = \frac{3}{16} \quad P'(1) = 3 \quad P''(0) = \frac{9}{32} \quad P''(1) = 18$$

Calculate the second moment of the distribution.

Exercise 2

The probability generating function for a discrete distribution satisfies the following:

$$P'(0) = \frac{3}{16} \quad P'(1) = 3 \quad P''(0) = \frac{9}{32} \quad P''(1) = 18$$

Calculate the second moment of the distribution.

$$\begin{aligned} \mathrm{E}[X] &= P'(1) = 3 \\ \mathrm{E}[X(X-1)] &= P''(1) = 18 \\ \mathrm{E}\left[X^2 - X\right] &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X] = 18 \\ \mathrm{E}\left[X^2\right] &= 18 + \mathrm{E}[X] = \boxed{21} \end{aligned}$$