A.1 Probability Review - Outline

A.1.4 Joint and Conditional Distributions

- Conditional Probability
- Joint Distributions
- Exercises

Conditional Probability

$$\mathrm{P}[AB] = \mathrm{P}[A] \cdot \mathrm{P}[B \mid A]$$

$$\mathrm{P}[B \mid A] = \frac{\mathrm{P}[AB]}{\mathrm{P}[A]}$$

$$\mathrm{P}[A \mid B] = \frac{\mathrm{P}[AB]}{\mathrm{P}[B]}$$

If $A_1, A_2, \ldots, A_n$ are a *partition* of the sample space, meaning $A_i \cap A_j = \emptyset$ for all $i \neq j$ and $\sum \mathrm{P}[A_i] = 1$,

Theorem (Law of Total Probability)

$$P[B] = \sum_j P[A_j] \cdot P[B \mid A_j]$$

Theorem (Bayes' Theorem)

$$P[A_i \mid B] = \frac{P[A_i B]}{P[B]} = \frac{P[A_i] \cdot P[B \mid A_i]}{\sum_j P[A_j] \cdot P[B \mid A_j]}$$

Example

Low risk individuals have losses that are exponential with mean 100, while high risk individuals have losses that are exponential with mean 200. You observe a loss that is greater than 300. If 70% of losses are from low risk individuals, what is the probability that the observed loss was from a low risk individual?

Let X = loss amount, L denote low risk, and H high risk.

$$\begin{aligned}
\mathrm{P}[\mathrm{L} \mid X > 300] &= \frac{\mathrm{P}[\mathrm{L}, X > 300]}{\mathrm{P}[X > 300]} \\
&= \frac{\mathrm{P}[\mathrm{L}] \cdot \mathrm{P}[X > 300 \mid \mathrm{L}]}{\mathrm{P}[\mathrm{L}] \cdot \mathrm{P}[X > 300 \mid \mathrm{L}] + \mathrm{P}[\mathrm{H}] \cdot \mathrm{P}[X > 300 \mid \mathrm{H}]} \\
&= \frac{0.7 \cdot e^{-300/100}}{0.7 \cdot e^{-300/100} + 0.3 \cdot e^{-300/200}} \\
&= \boxed{0.342}
\end{aligned}$$

Discrete Example

The joint distribution of X and Y is given by

		X		
		0	1	2
Y	1	0.1	0.2	0.3
	2	0.1	0.1	0.2

$$\mathrm{P}[X = 0] = 0.1 + 0.1 = 0.2$$

$$\mathrm{P}[X = 1] = 0.2 + 0.1 = 0.3$$

$$\mathrm{P}[X = 2] = 0.3 + 0.2 = 0.5$$

$$\mathrm{P}[Y = 1] = 0.1 + 0.2 + 0.3 = 0.6$$

$$\mathrm{P}[Y = 2] = 0.1 + 0.1 + 0.2 = 0.4$$

$$\mathrm{P}[X = 0 \mid Y = 1] = \frac{0.1}{0.6} = \frac{1}{6}$$

$$\mathrm{P}[X = 1 \mid Y = 1] = \frac{0.2}{0.6} = \frac{2}{6}$$

$$\mathrm{P}[X = 2 \mid Y = 1] = \frac{0.3}{0.6} = \frac{3}{6}$$

$$\mathrm{P}[Y = 1 \mid X = 2] = 0.6$$

$$\mathrm{P}[Y = 2 \mid X = 2] = 0.4$$

Formulas

Discrete: $$\mathrm{P}[X = x] = \sum_y \mathrm{P}[X = x, Y = y]$$

$$\mathrm{P}[X = x \mid Y = y] = \frac{\mathrm{P}[X = x, Y = y]}{\mathrm{P}[Y = y]} = \frac{\mathrm{P}[X = x, Y = y]}{\sum_x \mathrm{P}[X = x, Y = y]}$$

Continuous: $$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$f_{X|Y}(x \mid Y = y) = \frac{f(x, y)}{f_Y(y)} = \frac{f(x, y)}{\int_{-\infty}^{\infty} f(x, y)\, dx}$$

X and Y are independent if and only if 1) $f(x, y) = f(x) \cdot f(y)$ and 2) the range of X and Y is rectangular.

Exercise 1

A blood test indicates the presence of a particular disease 90% of the time when the disease is actually present. The same test indicates the presence of the disease 2% of the time when the disease is not present. One percent of the population actually has the disease. Calculate the probability that a person has the disease given that the test indicates the presence of the disease.

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Let T denote a positive test, and D the disease.

$$\begin{aligned}
\mathrm{P}[D \mid T] &= \frac{\mathrm{P}[DT]}{\mathrm{P}[T]} \\
&= \frac{\mathrm{P}[D] \cdot \mathrm{P}[T \mid D]}{\mathrm{P}[D] \cdot \mathrm{P}[T \mid D] + \mathrm{P}[D'] \cdot \mathrm{P}[T \mid D']} \\
&= \frac{0.01 \cdot 0.90}{0.01 \cdot 0.90 + 0.99 \cdot 0.02} \\
&= \boxed{0.3125}
\end{aligned}$$

Exercise 2

X and Y have joint density $0.3x^2y^{-3}$ for $0 < x < y < 10$. Find $\mathrm{P}[X \leq 3 \mid Y = 5]$.

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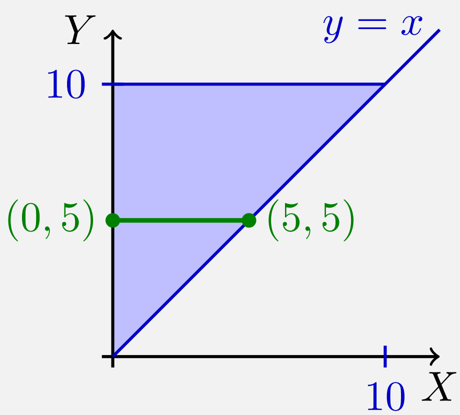

$$f_{X|Y}(x \mid Y = 5) = \frac{f(x, 5)}{\int_0^5 f(x, 5)dx}$$

$$= \frac{0.3x^2 5^{-3}}{\int_0^5 0.3x^2 5^{-3} dx}$$

$$= \frac{x^2}{\int_0^5 x^2\, dx} = \frac{3x^2}{125}$$

$$\mathrm{P}[X \leq 3 \mid Y = 5] = \int_0^3 \frac{3x^2}{125} dx$$

$$= \boxed{\frac{27}{125}}$$

Questions like this are very rare on the exam.