A.1 Probability Review - Outline

A.1.5 Conditional Moments

- Law of Total Expectation
- Double Expectation
- Exercises

Law of Total Expectation

For a discrete variable, $\mathrm{E}[g(X)] = \sum g(x) \cdot \mathrm{P}[X = x]$

If $A_1, \ldots, A_n$ are a partition of the sample space, then for any variable

$$\mathrm{E}[X] = \sum_i \mathrm{E}[X \mid A_i] \cdot \mathrm{P}[A_i]$$

$$\mathrm{E}[X^k] = \sum_i \mathrm{E}[X^k \mid A_i] \cdot \mathrm{P}[A_i]$$

Example: Low risk individuals have average annual losses of 100, high risk have average annual losses of 300.
If 40% of a group are low risk and 60% high risk, then the average for the group is $0.4 \cdot 100 + 0.6 \cdot 300 = 220$

MultiVariate Example

		X		
		0	1	2
Y	1	0.1	0.2	0.3
	2	0.1	0.1	0.2

$$\begin{aligned} \mathrm{E}[X] &= 0 \cdot 0.1 + 1 \cdot 0.2 + 2 \cdot 0.3 + 0 \cdot 0.1 + 1 \cdot 0.1 + 2 \cdot 0.2 \\ &= 0 \cdot 0.2 + 1 \cdot 0.3 + 2 \cdot 0.5 \\ &= 0 \cdot \mathrm{P}[X = 0] + 1 \cdot \mathrm{P}[X = 1] + 2 \cdot \mathrm{P}[X = 2] \\ &= 1.3 \end{aligned}$$

Last time, we saw

$$\mathrm{P}[X = 0 \mid Y = 1] = \frac{1}{6} \quad \mathrm{P}[X = 1 \mid Y = 1] = \frac{2}{6} \quad \mathrm{P}[X = 2 \mid Y = 1] = \frac{3}{6}$$

$$\mathrm{E}[X \mid Y = 1] = 0 \cdot \frac{1}{6} + 1 \cdot \frac{2}{6} + 2 \cdot \frac{3}{6} = \frac{4}{3}$$

Likewise, $\mathrm{E}[X \mid Y = 2] = \dfrac{5}{4}$

Example Continued

$$\mathrm{E}[X \mid Y = 1] = \frac{4}{3} \qquad \mathrm{E}[X \mid Y = 2] = \frac{5}{4}$$

Key point: $\mathrm{E}[X \mid Y]$ is a function of Y. *As a result, it is also a random variable.*

$$\begin{aligned} \mathrm{P}[\mathrm{E}[X \mid Y] = 4/3] &= \mathrm{P}[Y = 1] = 0.6 \\ \mathrm{P}[\mathrm{E}[X \mid Y] = 5/4] &= \mathrm{P}[Y = 2] = 0.4 \\ \mathrm{E}[\mathrm{E}[X \mid Y]] &= 0.6 \cdot \frac{4}{3} + 0.4 \cdot \frac{5}{4} \\ &= 1.3 \\ &= \mathrm{E}[X] \end{aligned}$$

That is not a coincidence.
Remember: $\mathrm{E}[X \mid Y]$ is random. $\mathrm{E}[X]$ is a (non-random) number.

Double Expectation

Recall that

$$\mathrm{E}[g(Y)] = \sum_y g(y) \cdot \mathrm{P}[Y = y]$$

$$\mathrm{E}[X] = \sum_i \mathrm{E}[X \mid A_i] \cdot \mathrm{P}[A_i]$$

$$\mathrm{E}[X] = \sum_y \mathrm{E}[X \mid Y = y] \cdot \mathrm{P}[Y = y]$$

$$= \mathrm{E}[\mathrm{E}[X \mid Y]]$$

This works for all raw moments:

$$\mathrm{E}[X^k] = \mathrm{E}[\mathrm{E}[X^k \mid Y]]$$

Central moments require an adjustment:

$$\mathrm{Var}[X] = \mathrm{E}[\mathrm{Var}[X \mid Y]] + \mathrm{Var}[\mathrm{E}[X \mid Y]]$$

Example

		X		
		0	1	2
Y	1	0.1	0.2	0.3
	2	0.1	0.1	0.2

Find $\mathrm{Var}[X]$

$$\mathrm{E}[X \mid Y = 1] = \frac{4}{3} \quad \mathrm{E}[X \mid Y = 2] = \frac{5}{4}$$

$$\mathrm{Var}[X \mid Y = 1] = 0^2 \cdot \frac{1}{6} + 1^2 \cdot \frac{2}{6} + 2^2 \cdot \frac{3}{6} - \left(\frac{4}{3}\right)^2 = \frac{5}{9}$$

$$\mathrm{Var}[X \mid Y = 2] = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{4} + 2^2 \cdot \frac{2}{4} - \left(\frac{5}{4}\right)^2 = \frac{11}{16}$$

$$\mathrm{Var}[X] = \mathrm{E}[\mathrm{Var}[X \mid Y]] + \mathrm{Var}[\mathrm{E}[X \mid Y]]$$

$$= 0.6 \cdot \frac{5}{9} + 0.4 \cdot \frac{11}{16} + \left(\frac{4}{3} - \frac{5}{4}\right)^2 \cdot 0.6 \cdot 0.4$$

$$= \boxed{0.61}$$

Exercise 1

I roll a fair six-sided die, and N comes up. I then flip N independent fair coins. Find the mean and variance of the number of heads.

Exercise 1

I roll a fair six-sided die, and N comes up. I then flip N independent fair coins. Find the mean and variance of the number of heads.

Let $H = \#$ of heads. If we knew N this would be easy.
That is a hint to use double expectation.

$$
\begin{aligned}
\mathrm{E}[H] &= \mathrm{E}[\mathrm{E}[H \mid N]] \\
&= \mathrm{E}\left[\frac{N}{2}\right] = \frac{1}{2} \cdot \mathrm{E}[N] \\
&= \frac{1}{2} \cdot \frac{1+6}{2} = \boxed{\frac{7}{4}} \\
\mathrm{Var}[H] &= \mathrm{E}[\mathrm{Var}[H \mid N]] + \mathrm{Var}[\mathrm{E}[H \mid N]] \\
&= \mathrm{E}\left[N \cdot \frac{1}{2} \cdot \frac{1}{2}\right] + \mathrm{Var}\left[\frac{N}{2}\right] \\
&= \frac{1}{4}\mathrm{E}[N] + \frac{1}{4}\mathrm{Var}[N] = \frac{1}{4} \cdot \frac{7}{2} + \frac{1}{4} \cdot \frac{35}{12} = \boxed{\frac{77}{48}}
\end{aligned}
$$

Exercise 2

The joint distribution of X and Y is as follows. Find $\mathrm{E}[Y]$ and $\mathrm{Var}[Y]$.

		X		
		0	1	2
Y	1	0.1	0.2	0.3
	2	0.1	0.1	0.2

Exercise 2

The joint distribution of X and Y is as follows. Find $\mathrm{E}[Y]$ and $\mathrm{Var}[Y]$.

		X		
		0	1	2
Y	1	0.1	0.2	0.3
	2	0.1	0.1	0.2

$$
\begin{aligned}
\mathrm{E}[Y] &= 1 \cdot (0.1 + 0.2 + 0.3) + 2 \cdot (0.1 + 0.1 + 0.2) \\
&= 1 \cdot 0.6 + 2 \cdot 0.4 = \boxed{1.4} \\
\mathrm{E}[Y^2] &= 1^2 \cdot 0.6 + 2^2 \cdot 0.4 = 2.2 \\
\mathrm{Var}[Y] &= 2.2 - 1.4^2 = \boxed{0.24} \\
\text{Or: } \mathrm{E}[Y \mid X = 0] &= 1 \cdot \frac{0.1}{0.1 + 0.1} + 2 \cdot \frac{0.1}{0.1 + 0.1} = 1.5 \\
\mathrm{E}[Y \mid X = 1] &= 1 \cdot \frac{0.2}{0.2 + 0.1} + 2 \cdot \frac{0.1}{0.2 + 0.1} = \frac{4}{3} \\
\mathrm{E}[Y \mid X = 2] &= 1 \cdot \frac{0.3}{0.3 + 0.2} + 2 \cdot \frac{0.2}{0.3 + 0.2} = 1.4 \\
\mathrm{E}[Y] &= \mathrm{E}[\mathrm{E}[Y \mid X]] = 1.5 \cdot 0.2 + \frac{4}{3} \cdot 0.3 + 1.4 \cdot 0.5 = \boxed{1.4}
\end{aligned}
$$

Exercise 2 (Continued)

		X		
		0	1	2
Y	1	0.1	0.2	0.3
	2	0.1	0.1	0.2

$$\mathrm{E}[Y \mid X = 0] = 1.5$$
$$\mathrm{E}[Y \mid X = 1] = 4/3$$
$$\mathrm{E}[Y \mid X = 2] = 1.4$$

$$\begin{aligned}
\mathrm{Var}[Y \mid X = 0] &= (2-1)^2 \cdot 0.5 \cdot 0.5 = 0.25 \\
\mathrm{Var}[Y \mid X = 1] &= (2-1)^2 \cdot \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9} \\
\mathrm{Var}[Y \mid X = 2] &= (2-1)^2 \cdot 0.6 \cdot 0.4 = 0.24 \\
\mathrm{Var}[Y] &= \mathrm{E}[\mathrm{Var}[Y \mid X]] + \mathrm{Var}[\mathrm{E}[Y \mid X]] \\
&= 0.2 \cdot 0.25 + 0.3 \cdot \frac{2}{9} + 0.5 \cdot 0.24 \\
&\quad + \left(1.5^2 \cdot 0.2 + \frac{4^2}{3^2} \cdot 0.3 + 1.4^2 \cdot 0.5\right) - 1.4^2 \\
&= \boxed{0.24}
\end{aligned}$$