A.1 Probability Review - Outline

A.1.6 Mixtures
- Example
- Definitions
- Examples
- Exercises

Example 1

An insurance company has two types of customers: high risk customers, whose annual losses are exponential with mean 10, and low risk customers, whose annual losses are exponential with mean 1. 30% of the customers are high risk.
A customer is chosen at random. What is the probability that their annual loss is at least 5?

Let L denote low risk, H high risk, X annual losses.

$$\begin{aligned} \mathrm{P}[X \geq 5] &= \mathrm{P}[L, X \geq 5] + \mathrm{P}[H, X \geq 5] \\ &= \mathrm{P}[L] \cdot \mathrm{P}[X \geq 5 \mid L] + \mathrm{P}[H] \cdot \mathrm{P}[X \geq 5 \mid H] \\ &= 0.7 \cdot e^{-5/1} + 0.3 \cdot e^{-5/10} \\ &= \boxed{0.187} \end{aligned}$$

n-point Mixture

Suppose we have n cases.

Exactly 1 case occurs.

Case i: $Y = X_i$, $\mathrm{P}[\text{case } i] = a_i$, $\sum a_i = 1$

$$\begin{aligned}
F_Y(y) &= a_1 F_{X_1}(y) + \cdots + a_n F_{X_n}(y) \\
f_Y(y) &= a_1 f_{X_1}(y) + \cdots + a_n f_{X_n}(y) \\
\mathrm{E}[Y] &= \mathrm{E}[\mathrm{E}[Y \mid \text{case}]] = a_1 \mathrm{E}[X_1] + \cdots + a_n \mathrm{E}[X_n] \\
\mathrm{E}[Y^k] &= a_1 \mathrm{E}[X_1^k] + \cdots + a_n \mathrm{E}[X_n^k] \\
\mathrm{Var}[Y] &= \mathrm{E}[\mathrm{Var}[Y \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[Y \mid \text{Case}]] \\
&= \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2
\end{aligned}$$

Warning: $Y \neq \sum a_i X_i$
We will discuss the differences next lesson.

Example 2

An insurance company has two types of customers: high risk, whose annual losses have mean 10 and variance 50, and low risk, whose annual losses have mean 5 and variance 20. 30% of the customers are high risk.
Find the mean and variance of the annual loss of a randomly chosen customer.

Let Y be a random annual loss, L = low risk, H = high risk.

$$\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[\mathrm{E}[Y \mid \text{Case}]] \\
&= \mathrm{P}[H] \cdot \mathrm{E}[Y \mid H] + \mathrm{P}[L] \cdot \mathrm{E}[Y \mid L] \\
&= 0.3 \cdot 10 + 0.7 \cdot 5 = \boxed{6.5} \\
\mathrm{Var}[Y] &= \mathrm{E}[\mathrm{Var}[Y \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[Y \mid \text{Case}]] \\
&= 0.3 \cdot 50 + 0.7 \cdot 20 + (10-5)^2 \cdot 0.3 \cdot 0.7 \\
&= \boxed{34.25}
\end{aligned}$$

Example 2 (Continued)

$$\mathrm{P}[H] = 0.3 \qquad \mathrm{P}[L] = 0.7$$
$$\mathrm{E}[Y \mid H] = 10 \qquad \mathrm{Var}[Y \mid H] = 50$$
$$\mathrm{E}[Y \mid L] = 5 \qquad \mathrm{Var}[Y \mid L] = 20$$
$$\mathrm{E}[Y] = 6.5$$

Or using raw moments:

$$\begin{aligned}
\mathrm{E}[Y^2] &= \mathrm{E}[\mathrm{E}[Y^2 \mid \mathrm{Case}]] \\
&= \mathrm{P}[H] \cdot \mathrm{E}[Y^2 \mid H] + \mathrm{P}[L] \cdot \mathrm{E}[Y^2 \mid L] \\
&= \mathrm{P}[H] \cdot \left(\mathrm{Var}[Y \mid H] + (\mathrm{E}[Y \mid H])^2\right) \\
&\quad + \mathrm{P}[L] \cdot \left(\mathrm{Var}[Y \mid L] + (\mathrm{E}[Y \mid L])^2\right) \\
&= 0.3(50 + 10^2) + 0.7(20 + 5^2) \\
&= 76.5 \\
\mathrm{Var}[Y] &= 76.5 - 6.5^2 = \boxed{34.25}
\end{aligned}$$

Example 3

An insurance company has 3 types of customers, whose annual losses are described below. Find the mean and variance of the annual loss of a randomly chosen customer.

Type	Proportion of Customers	Average Annual Loss	Variance of Annual Losses
Low	60%	5	20
Med	30%	10	50
High	10%	30	100

Y = random annual loss, L = low risk, M = med. risk, H = high risk.

$$\begin{aligned}
\mathrm{E}[Y] &= 0.6 \cdot \mathrm{E}[Y \mid L] + 0.3 \cdot \mathrm{E}[Y \mid M] + 0.1 \cdot \mathrm{E}[Y \mid H] \\
&= 0.6 \cdot 5 + 0.3 \cdot 10 + 0.1 \cdot 30 = \boxed{9} \\
\mathrm{E}[\mathrm{Var}[Y \mid \mathrm{Case}]] &= 0.6 \cdot 20 + 0.3 \cdot 50 + 0.1 \cdot 100 = 37 \\
\mathrm{Var}[\mathrm{E}[Y \mid \mathrm{Case}]] &= 5^2 \cdot 0.6 + 10^2 \cdot 0.3 + 30^2 \cdot 0.1 \\
&\quad - (5 \cdot 0.6 + 10 \cdot 0.3 + 30 \cdot 0.1)^2 \\
&= 135 - 9^2 = 54 \\
\mathrm{Var}[Y] &= 37 + 54 = \boxed{91}
\end{aligned}$$

Exercise 3: Raw Moments

Type	Proportion of Customers	Average Annual Loss	Variance of Annual Losses
Low	60%	5	20
Med	30%	10	50
High	10%	30	100

Or with raw moments:

$$\begin{aligned}
\mathrm{E}[Y] &= 0.6 \cdot \mathrm{E}[Y \mid L] + 0.3 \cdot \mathrm{E}[Y \mid M] + 0.1 \cdot \mathrm{E}[Y \mid H] \\
&= 0.6 \cdot 5 + 0.3 \cdot 10 + 0.1 \cdot 30 = \boxed{9} \\
\mathrm{E}[Y^2] &= 0.6 \cdot \mathrm{E}[Y^2 \mid L] + 0.3 \cdot \mathrm{E}[Y^2 \mid M] + 0.1^2 \cdot \mathrm{E}[Y^2 \mid H] \\
&= 0.6(20 + 5^2) + 0.3(50 + 10^2) + 0.1(100 + 30^2) \\
&= 172 \\
\mathrm{Var}[Y] &= 172 - 9^2 = \boxed{91}
\end{aligned}$$

Exercise 1

An insurance company has two types of customers: high risk customers, whose loss amounts have mean 10 and variance 50, and low risk customers, whose losses have mean 5 and variance 20. The expected amount of a randomly chosen loss is 8.
Find the variance of a randomly chosen loss.

Exercise 1

An insurance company has two types of customers: high risk customers, whose loss amounts have mean 10 and variance 50, and low risk customers, whose losses have mean 5 and variance 20. The expected amount of a randomly chosen loss is 8.
Find the variance of a randomly chosen loss.

Y = random loss, L = low risk, H = high risk. Let $p = \mathrm{P}[L]$.

$$\begin{aligned} \mathrm{E}[Y] &= 8 = p \cdot 5 + (1-p) \cdot 10 \\ p &= 0.4 \\ \mathrm{Var}[Y] &= \mathrm{E}[\mathrm{Var}[Y \mid \text{Case}]] + \mathrm{Var}[\mathrm{E}[Y \mid \text{Case}]] \\ &= 0.4 \cdot 20 + 0.6 \cdot 50 + (10-5)^2 \cdot 0.4 \cdot 0.6 \\ &= \boxed{44} \end{aligned}$$

Exercise 2

Rural losses have hazard rate $h(x) = 0.2x$ for $x > 0$, while urban losses have hazard rate $h(x) = 0.03x^2$ for $x > 0$. If 30% of losses are rural, what proportion of losses are at most 5?

Exercise 2

Rural losses have hazard rate $h(x) = 0.2x$ for $x > 0$, while urban losses have hazard rate $h(x) = 0.03x^2$ for $x > 0$. If 30% of losses are rural, what proportion of losses are at most 5?

Let R denote rural, U urban, and Y a random loss.

$$\begin{aligned}
\mathrm{P}[Y \le 5] &= \mathrm{P}[R] \cdot \mathrm{P}[Y \le 5 \mid R] + \mathrm{P}[U] \cdot \mathrm{P}[Y \le 5 \mid U] \\
&= 0.3 \cdot \mathrm{P}[Y \le 5 \mid R] + 0.7 \cdot \mathrm{P}[Y \le 5 \mid U] \\
\mathrm{P}[Y \le 5 \mid R] &= 1 - \mathrm{P}[Y > 5 \mid R] \\
&= 1 - e^{-\int_0^5 0.2x\,dx} = 1 - e^{-0.1 \cdot 5^2} = 0.918 \\
\mathrm{P}[Y \le 5 \mid U] &= 1 - e^{-\int_0^5 0.03x^2\,dx} \\
&= 1 - e^{-0.01 \cdot 5^3} = 0.713 \\
\mathrm{P}[Y \le 5] &= 0.3 \cdot 0.918 + 0.7 \cdot 0.713 = \boxed{0.775}
\end{aligned}$$