The Infinite Actuary Exam STAM Online Course
A.2.3. Pareto Distributions
Last updated April 11, 2018

1. [3.F01.37] For watches produced by a certain manufacturer:

   (i) Lifetimes follow a single-parameter Pareto distribution with $\alpha > 1$ and $\theta = 4$.
   (ii) The expected lifetime of a watch is 8 years.

   Calculate the probability that the lifetime of a watch is at least 6 years.

   A. 0.44 B. 0.50 C. 0.56 D. 0.61 E. 0.67

From the tables, $\mathrm{E}[X^k] = \dfrac{\alpha\theta^k}{\alpha - k}$ so $8 = \dfrac{4\alpha}{\alpha - 1}$ and thus $\alpha = 2$.
We want $\mathrm{P}[X > 6] = S(6) = 1 - F(6) = 1 - \left(1 - \left(\frac{4}{6}\right)^2\right) = \left(\frac{2}{3}\right)^2 = \boxed{\frac{4}{9}}$

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2. [3.S06.25] Calculate the skewness of a Pareto distribution with $\alpha = 4$ and $\theta = 1{,}000$.

   A. Less than 2
   B. At least 2, but less than 4
   C. At least 4, but less than 6
   D. At least 6, but less than 8
   E. At least 8

The raw moments are in the tables, giving us

$$\begin{aligned}
\mu &= \frac{\theta}{\alpha - 1} = 333.3 \\
\mu_2' &= \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = 333{,}333.3 \\
\mu_3' &= \frac{6\theta^3}{(\alpha-1)(\alpha-2)(\alpha-3)} = 10^9
\end{aligned}$$

Therefore $\sigma^3 = \mathrm{Var}(X)^{3/2} = \left(E(X^2) - \mu^2\right)^{3/2} = (222{,}222)^{1.5} = 104{,}756{,}560$ and

$$\begin{aligned}
\mu_3 &= \mathrm{E}\left[(X-\mu)^3\right] = \mathrm{E}[X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3] \\
&= \mathrm{E}[X^3] - 3\mu\mathrm{E}[X^2] + 3\mu^2\mu - \mu^3 \\
&= 740{,}740{,}741
\end{aligned}$$

and so the skewness is $740.740/104.757 = \boxed{7.07}$

Later on, we will see that the answer does't depend on θ, so you can set $\theta = 1$ to make the numbers easier to work with.

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3. Losses are a 2-point mixture. 30% of the time, losses come from a Pareto distribution with $\alpha = 3$ and $\theta = 10$, and 70% of the time losses come from a Pareto distribution with $\alpha = 6$ and $\theta = 10$. What is the median loss amount?

A. 1.50 B. 1.53 C. 1.57 D. 1.60 E. 1.64

$$F(x) = 0.3\left[1 - \left(\frac{10}{x+10}\right)^3\right] + 0.7\left[1 - \left(\frac{10}{x+10}\right)^6\right]$$

$$0.5 = 0.3 - 0.3u + 0.7 - 0.7u^2 \quad \text{where } u = \left(\frac{10}{x+10}\right)^3$$

$$0.7u^2 + 0.3u - 0.5 = 0$$

$$u = \left(\frac{10}{x+10}\right)^3 = 0.6576$$

$$x = \boxed{1.5}$$

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4. [M.S05.9] A loss, X, follows a 2-parameter Pareto distribution with $\alpha = 2$ and unspecified parameter θ. You are given:

$$\mathrm{E}[X - 100 \mid X > 100] = \frac{5}{3}\mathrm{E}[X - 50 \mid X > 50]$$

Calculate $\mathrm{E}[X - 150 \mid X > 150]$.

A. 150 B. 175 C. 200 D. 225 E. 250

For a Pareto distribution, the distribution of $(X - d \mid X > d)$ is a Pareto with the same α and with an updated $\theta' = \theta + d$. So we get

$$\frac{\theta + 100}{2-1} = \frac{5}{3}\frac{\theta + 50}{2-1}$$

$$\theta = 25$$

and then $\mathrm{E}[X - 150 \mid X > 150] = (25 + 150)/(2-1) = \boxed{175}$

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5. [M.S05.34] The distribution of a loss, X , is a two-point mixture:

(i) With probability 0.8, X has a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 100$.

(ii) With probability 0.2, X has a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 3000$.

Calculate $\Pr(X \leq 200)$.

A. 0.76 B. 0.79 C. 0.82 D. 0.85 E. 0.88

$$F(200) = 0.8\left[1 - \left(\frac{100}{100+200}\right)^2\right] + 0.2\left[1 - \left(\frac{3{,}000}{3{,}000+200}\right)^4\right]$$

$$= 0.711 + 0.046 = \boxed{0.757}$$

Suppose that X is a mixture of two distributions, one of which is a single parameter Pareto distribution with $\alpha = 3$ and $\theta = 100$ and the other is a single parameter Pareto with $\alpha = 3$ and $\theta = 100 + \delta$ for some value of δ.

If $\mathrm{E}[X] = 165$ and $\mathrm{E}\left(X^2\right) = 37{,}200$, then what is δ? 1020304050 Let p be the probability that we are selecting from the distribution with $\theta = 100 + \delta$. Then

$$\begin{aligned}
\mathrm{E}[X] &= (1-p)\frac{3 \cdot 100}{3-1} + p\frac{3(100+\delta)}{3-1} \\
165 &= \frac{3}{2}(100 - 100p + 100p + \delta p) = 150 + 1.5\delta p \\
10 &= \delta p
\end{aligned}$$

and for the second moment,

$$\begin{aligned}
\mathrm{E}[X^2] = 37{,}200 &= (1-p)\frac{3 \cdot 100^2}{3-2} + p\frac{3(100+\delta)^2}{3-2} \\
12{,}400 &= (1-p)100^2 + p \cdot 100^2 + 2p \cdot 100\delta + p\delta^2 \\
2{,}400 &= 200(10) + 10(\delta) \qquad \text{by plugging in } 10 = p\delta \\
\delta &= \boxed{40}
\end{aligned}$$

6. Based on [4B.F99.19] Suppose that X has a Pareto distribution with $\alpha = 2$ and $\theta = 10,000$, and Y has a Burr distribution with parameters $\alpha = 1, \gamma = 2$ and $\theta = \sqrt{20,000}$.

   Let r be the ratio of $\mathrm{P}[X > d]$ to $\mathrm{P}[Y > d]$. What is $\lim\limits_{d \to \infty} r$?

   A. 0 B. 0.25 C. 1 D. 5,000 E. ∞

$$\begin{aligned}
\mathrm{P}[X > d] &= 1 - F_X(d) = \left(\frac{\theta}{d+\theta}\right)^\alpha = \left(\frac{10{,}000}{d + 10{,}000}\right)^2 \\
\mathrm{P}[Y > d] &= u^\alpha = \frac{1}{1 + \left(\frac{d}{\theta}\right)^2} = \frac{20{,}000}{20{,}000 + d^2} \\
r &= \frac{\left(\dfrac{10{,}000}{d + 10{,}000}\right)^2}{\dfrac{20{,}000}{20{,}000 + d^2}} \\
\lim_{d \to \infty} r &= \lim_{d \to \infty} \frac{\dfrac{10{,}000^2}{d^2}}{\dfrac{20{,}000}{d^2}} = \frac{10{,}000^2}{20{,}000} = \boxed{5{,}000}
\end{aligned}$$
