A.2 Key Distributions - Outline

A.2.3 Pareto Distribution

- Pareto
- Single Parameter Pareto
- Exercises

Pareto

"Pareto" by itself refers to the 2-parameter Pareto (A.2.3.1 on the formula sheet).

$$\mathrm{P}[X > x] = \left(\frac{\theta}{x+\theta}\right)^{\alpha} \quad x > 0$$

$$\mathrm{E}[X] = \frac{\theta}{\alpha - 1} \quad \text{if } \alpha > 1$$

$$\mathrm{E}[X^2] = \frac{2\theta^2}{(\alpha-1)(\alpha-2)} \quad \text{if } \alpha > 2$$

$$\begin{aligned}\mathrm{Var}[X] &= \left(\frac{\theta}{\alpha-1}\right)^2 \cdot \frac{\alpha}{\alpha-2} \\ &= (\mathrm{E}[X])^2 \cdot \frac{\alpha}{\alpha-2}\end{aligned}$$

Note that $\alpha/(\alpha-2) \to 1$ as $\alpha \to \infty$ and $\alpha/(\alpha-2) \to \infty$ as $\alpha \downarrow 2$

Pareto

The Pareto isn't quite memoryless:

$$\begin{aligned}
\mathrm{P}[X-d>x \mid X>d] &= \frac{\mathrm{P}[X>x+d]}{\mathrm{P}[X>d]} \\
&= \frac{\left(\dfrac{\theta}{x+d+\theta}\right)^{\alpha}}{\left(\dfrac{\theta}{d+\theta}\right)^{\alpha}} \\
&= \left(\frac{d+\theta}{x+d+\theta}\right)^{\alpha}
\end{aligned}$$

i.e., $(X-d \mid X>d)$ is Pareto with the same α and a new $\theta' = \theta + d$.

So $\mathrm{E}[X-d \mid X>d] = \dfrac{\theta'}{\alpha-1} = \dfrac{\theta+d}{\alpha-1}$

and $\mathrm{Var}[X-d \mid X>d] = \left(\dfrac{\theta+d}{\alpha-1}\right)^2 \cdot \dfrac{\alpha}{\alpha-2}$

Single Parameter Pareto

The "single parameter" Pareto has range $X > \theta$ and density

$$f(x) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}} \quad x > \theta$$

A single parameter Pareto is a Pareto distribution shifted by θ.

$$\begin{aligned}
\mathrm{E}[X] &= \frac{\theta}{\alpha-1} + \theta = \frac{\theta + (\alpha-1)\theta}{\alpha-1} \\
&= \frac{\alpha\theta}{\alpha-1} \quad \text{(in tables)} \\
\mathrm{Var}[X] &= \mathrm{Var}[\text{Pareto}] \\
&= \left(\frac{\theta}{\alpha-1}\right)^2 \cdot \frac{\alpha}{\alpha-2}
\end{aligned}$$

It is unlikely that memorizing the variance will be useful.

Exercise 1

X is Pareto distributed with $\mathrm{E}[X] = 5$ and $\mathrm{E}[X^2] = 100$.
Find $\mathrm{P}[X \leq 20 \mid X > 10]$.

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$$\begin{aligned}
\mathrm{E}[X] &= \frac{\theta}{\alpha - 1} = 5 \\
\mathrm{E}[X^2] &= \frac{2\theta^2}{(\alpha-1)(\alpha-2)} = 100 \\
\frac{\mathrm{E}[X^2]}{(\mathrm{E}[X])^2} &= \frac{2\theta^2}{(\alpha-1)(\alpha-2)} \cdot \frac{(\alpha-1)^2}{\theta^2} \\
\frac{100}{5^2} &= \frac{2(\alpha-1)}{\alpha-2} \\
4(\alpha-2) &= 2\alpha - 2 \\
\alpha &= 3 \\
\theta &= 10
\end{aligned}$$

Exercise 1 Cont.

$\alpha = 3, \theta = 10$, want $\mathrm{P}[X \leq 20 \mid X > 10]$

$$\begin{aligned}
\mathrm{P}[X \leq 20 \mid X > 10] &= \frac{\mathrm{P}[X \leq 20] - \mathrm{P}[X \leq 10]}{\mathrm{P}[X > 10]} \\
&= \frac{\mathrm{P}[X > 10] - \mathrm{P}[X > 20]}{\mathrm{P}[X > 10]} \\
&= \frac{\left(\frac{10}{10+10}\right)^3 - \left(\frac{10}{20+10}\right)^3}{\left(\frac{10}{10+10}\right)^3} \\
&= \boxed{\frac{19}{27}}
\end{aligned}$$

Exercise 2

A portfolio consists of 30 liability risks and 20 property risks, all with identical claim count distributions. Loss sizes for liability risks have a Pareto distribution with $\alpha = 4$ and $\theta = 30$. Loss sizes for property risks have a Pareto distribution with $\alpha = 3$ and $\theta = 50$. Find the variance of the loss size for this portfolio for a single claim.

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$$\mathrm{Var}[X] = \mathrm{E}[\mathrm{Var}(X \mid \mathrm{Case})] + \mathrm{Var}[\mathrm{E}(X \mid \mathrm{Case})]$$

$$\mathrm{E}[X \mid \text{liability}] = \frac{30}{4-1} = 10 \quad \mathrm{E}[X \mid \text{property}] = \frac{50}{3-1} = 25$$

$$\mathrm{Var}[X \mid \text{liability}] = 10^2 \cdot \frac{4}{4-2} = 200$$

$$\mathrm{Var}[X \mid \text{property}] = 25^2 \cdot \frac{3}{3-2} = 1{,}875$$

$$\begin{aligned}\mathrm{Var}[X] &= \left[\frac{30}{50} \cdot 200 + \frac{20}{50} \cdot 1{,}875\right] + (25-10)^2 \cdot 0.6 \cdot 0.4 \\ &= \boxed{924}\end{aligned}$$