A.3 Severity Models - Outline

A.3.1 Scale Parameters
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Scale Parameters

Suppose the distribution of X depends on parameters θ, α, β etc.

θ is a *scale parameter* if the distribution of cX is the same type of distribution as X, with all the same parameter values, except for a new $\theta' = c\theta$.

Examples

If $X \sim \text{Gamma}(\alpha = 3, \theta = 100)$ then
$2X \sim \text{Gamma}(\alpha = 3, \theta' = 200)$.

If $X \sim \text{Pareto}(\alpha = 4, \theta = 50)$ then $5X \sim \text{Pareto}(\alpha = 4, \theta' = 250)$.

On the exam tables, θ (and only θ) is always a scale parameter (except for inverse Gaussian).

Scale Parameters: Non-examples

Scaling Normals

If $X \sim N(\mu, \sigma^2)$ then $\mathrm{E}[cX] = c \cdot \mathrm{E}[X]$ and $\mathrm{Var}[cX] = c^2\mathrm{Var}[X]$,
$cX \sim N(\mu' = c\mu, (\sigma')^2 = c^2\sigma^2)$.

μ isn't a scale parameter because σ^2 changes too.

Scaling Lognormals

If $X \sim LN(\mu, \sigma^2)$, then $\ln(X) = Y \sim N(\mu, \sigma^2)$
$\ln(cX) = \ln(X) + \ln(c) \sim N(\mu, \sigma^2) + \ln(c) = N(\mu + \ln(c), \sigma^2)$
$cX \sim LN(\mu + \ln(c), \sigma^2)$.

Only μ changes.
But μ isn't a scale parameter because we add $\ln(c)$ instead of multiplying by c.

Inflation

Suppose that X is a loss amount this year. If there is inflation of i, then after inflation losses have the same distribution as $(1 + i)X$.

If θ is a scale parameter, then after inflation $\theta' = (1 + i)\theta$.

After n years of inflation, $\theta' = (1 + i)^n \cdot \theta$.

If $X \sim LN(\mu, \sigma^2)$ then $cX \sim LN(\mu' = \mu + \ln(c), \sigma^2)$
With n years of inflation,
$\mu' = \mu + \ln[(1 + i)^n] = \mu + n \cdot \ln(1 + i)$ and σ^2 is unchanged.

We can also work with $(1 + i)^n X$ instead of adjusting the parameters.

Example

Claim severity has a Pareto distribution with mean \$25,000 and $\alpha = 3$. If inflation increases all claims by 20%, what is the resulting probability of a claim exceeding \$50,000?
First, let's find the old θ.

$$\mathrm{E}[X] = 25{,}000 = \frac{\theta}{\alpha - 1}$$
$$\theta = (3-1) \cdot 25{,}000 = 50{,}000$$

After inflation, $\theta' = 1.2\theta = 60{,}000$

$$S_{new}(50{,}000) = \left(\frac{60{,}000}{50{,}000 + 60{,}000}\right)^3 = \boxed{0.162}$$

$$\text{Or: } \mathrm{P}[1.2X > 50{,}000] = \mathrm{P}[X > 41{,}667] = \left(\frac{50{,}000}{41{,}667 + 50{,}000}\right)^3 = \boxed{0.162}$$

Exercise 1

Monthly expenses are independent exponentials, each with median 100. Expenses next year are 10% higher due to regulatory changes. What is the variance of next year's total expenses?

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$$0.5 = F(x) = 1 - e^{-100/\theta}$$
$$\theta = -100/\ln(0.5) \approx 144.3$$
$$\theta' = 1.1 \cdot \theta \approx 158.7$$

Next year's expenses $Y \sim \text{Gamma}(\alpha = 12, \theta' = 158.7)$

$$12(\theta')^2 = \boxed{302{,}215}$$

Or: This year's expenses $X \sim \text{Gamma}(\alpha = 12, \theta = 144.3)$

$$\text{Var}[X] = 12\theta^2 = 249{,}764$$
$$\text{Var}[1.1X] = 1.1^2 \cdot 249{,}764 = \boxed{302{,}215}$$

Exercise 2

Losses in 2013 were lognormal$(\mu = 3, \sigma^2 = 1.2)$. Losses increase by 5% per year due to inflation. What is the probability that a randomly selected loss in 2018 will exceed 20?

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Losses in 2013 were lognormal($\mu = 3, \sigma^2 = 1.2$). Losses increase by 5% per year due to inflation. What is the probability that a randomly selected loss in 2018 will exceed 20?
Let $X \sim LN(\mu = 3, \sigma^2 = 1.2)$.

$$\begin{aligned}
\mathrm{P}[1.05^5 \cdot X > 20] &= \mathrm{P}[X > 15.67] = \mathrm{P}[\ln(X) > \ln(15.67)] \\
&= 1 - \Phi\left(\frac{\ln(15.67) - 3}{\sqrt{1.2}}\right) \\
&= 1 - \Phi(-0.23) = \Phi(0.23) = \boxed{0.5910} \\
\text{Or: } (1.05)^5 X &\sim LN(3 + 5\ln(1.05), 1.2) \\
\mathrm{P}[1.05^5 X > 20] &= \mathrm{P}[N(3.244, 1.2) > \ln(20)] \\
&= 1 - \Phi\left(\frac{\ln(20) - 3.244}{\sqrt{1.2}}\right) \\
&= 1 - \Phi(-0.23) = \boxed{0.5910}
\end{aligned}$$