0.1 Probability Review - Outline

0.1.2 Severity Distributions
- Density and CDF Distributions
- Moments
- Variances
- Exercises

Cumulative Distribution Function (CDF)

The cumulative distribution function (CDF) of X is

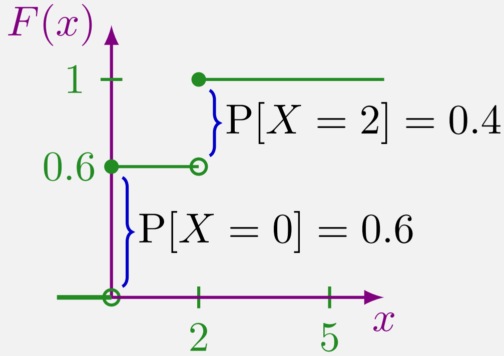
$$F(x) = \mathrm{P}[X \leq x]$$

$$F(\infty) = \lim_{x \to \infty} \mathrm{P}[X \leq x] = 1$$

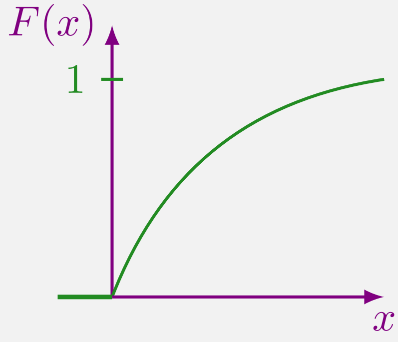
$$F(-\infty) = 0$$

$$S(x) = 1 - F(x) = \mathrm{P}[X > x] = \text{survival function}$$

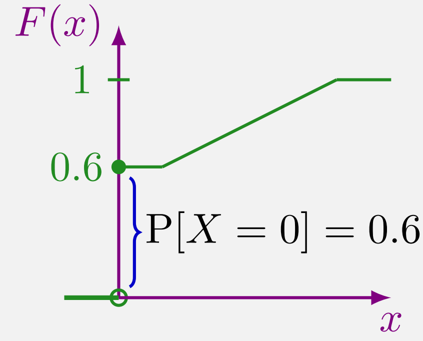
X is a *loss variable* if $X \geq 0$, in which case $F(x) = 0$ for all $x < 0$.



Discrete



Continuous



Mixed

Densities

X is a continuous variable if $F(x)$ is continuous. For us, this will also imply that $F(x)$ is differentiable. The density $f(x)$ is given by

$$
\begin{aligned}
f(x) &= F'(x) \\
\mathrm{P}[a < X \leq b] &= \mathrm{P}[X \leq b] - \mathrm{P}[X \leq a] = F(b) - F(a) \\
&= \int_a^b f(x)dx \\
1 &= \int_{-\infty}^{\infty} f(x)dx \\
f(x)\,dx &\text{ ``='' } \mathrm{P}[x < X \leq x + dx] \\
0 &\leq f(x)
\end{aligned}
$$

There is no upper limit on $f(x)$. In particular, $f(x)$ is not a probability, and can be greater than 1. For example, if X is uniform on $(0, 0.1)$ then $f(x) = 1/(0.1 - 0) = 10$

Moments

If X is discrete:	If X is continuous
$\mathrm{E}[X] = \sum_x x \cdot \mathrm{P}[X = x]$	$\mathrm{E}[X] = \int x \cdot f(x)dx$
$\mathrm{E}\left[X^k\right] = \sum_x x^k \cdot \mathrm{P}[X = x]$	$\mathrm{E}\left[X^k\right] = \int x^k \cdot f(x)dx$
$\mathrm{E}[g(X)] = \sum_x g(x) \cdot \mathrm{P}[X = x]$	$\mathrm{E}[g(X)] = \int g(x) \cdot f(x)dx$

For a mixed distribution:

- Use discrete formula over discrete values
- Use continuous formula over continuous values
- Sum the two pieces

Survival Function Approach

Recall that $S(x) = \mathrm{P}[X > x] = 1 - F(x)$
$S'(x) = -F'(x) = -f(x)$
For any loss variable X, integration by parts with $u = x$ and $v = -S(x)$ gives

$$\begin{aligned}
\mathrm{E}[X] &= \int_0^\infty x \cdot f(x)\,dx \\
&= x \cdot [-S(x)]\Big|_0^\infty + \int_0^\infty S(x)dx \\
&= 0 + \int_0^\infty S(x)dx
\end{aligned}$$

Also true, but less useful: If $g(0) = 0$,

$$\mathrm{E}[g(X)] = \int_0^\infty g'(x) \cdot S(x)dx$$

Variance

$$\begin{aligned}
\mu &= \mathrm{E}[X] \\
\sigma^2 &= \mathrm{Var}[X] \\
&= \mathrm{E}\big[(X-\mu)^2\big] \\
&= \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \\
\sigma &= \mathrm{SD}[X]
\end{aligned}$$

Bernoulli shortcut: If $\mathrm{P}[X = a] + \mathrm{P}[X = b] = 1$ then

$$\mathrm{Var}[X] = (b-a)^2 \cdot \mathrm{P}[X = a] \cdot \mathrm{P}[X = b]$$

$$\text{If } Y = cX \text{ then } \mathrm{E}[Y^k] = \mathrm{E}[(cX)^k] = c^k\mathrm{E}[X^k]$$

$$\mathrm{Var}[cX] = c^2 \cdot \mathrm{Var}[X]$$

$$\mathrm{SD}[cX] = |c| \cdot \mathrm{SD}[X]$$

Coefficient of Variation

Definition (Coefficient of Variation)

$\mathrm{CV}[X] = \dfrac{\sigma}{\mu}$

Note that for $c > 0$,

$$\begin{aligned}\mathrm{CV}[X] &= \frac{\sigma}{\mu} = \frac{\mathrm{SD}[X]}{\mathrm{E}[X]} \\ \mathrm{CV}[cX] &= \frac{\mathrm{SD}[cX]}{\mathrm{E}[cX]} = \frac{c\mathrm{SD}[X]}{c\mathrm{E}[X]} \\ &= \mathrm{CV}[X]\end{aligned}$$

Exercise 1

$\mathrm{P}[X = 100] = 0.3$ and $\mathrm{P}[X = 300] = 0.7$. Find the coefficient of variation of X.

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P$[X = 100] = 0.3$ and P$[X = 300] = 0.7$. Find the coefficient of variation of X.

$$\begin{aligned}
\mu = \mathrm{E}[X] &= 0.3 \cdot 100 + 0.7 \cdot 300 = 240 \\
\mathrm{E}[X^2] &= 0.3 \cdot 100^2 + 0.7 \cdot 300^2 = 66{,}000 \\
\sigma^2 = \mathrm{Var}[X] &= \mathrm{E}[X^2] - (\mathrm{E}[X])^2 \\
&= 66{,}000 - 240^2 = 8{,}400 \\
\text{Or: } \mathrm{E}[(X-\mu)^2] &= 0.3 \cdot (100 - 240)^2 + 0.7 \cdot (300 - 240)^2 \\
&= 8{,}400 \\
\text{Or Bernoulli Shortcut: } &= (300 - 100)^2 \cdot 0.3 \cdot 0.7 = 8{,}400 \\
\mathrm{CV}[X] &= \frac{\sigma}{\mu} = \frac{\sqrt{8{,}400}}{240} = \boxed{0.382}
\end{aligned}$$

Exercise 2

X has survival function $S(x) = 4^3/x^3$ for $x \geq 4$. Find E$[X]$.

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X has survival function $S(x) = 4^3/x^3$ for $x \geq 4$. Find $\mathrm{E}[X]$.

From the formula given, $S(4) = 4^3/4^3 = 1$. But $S(x) \leq 1$ because it is a probability, and $S(x)$ is decreasing, so $S(x) = 1$ for $x < 4$

$$S(x) = \begin{cases} 1 & x < 4 \\ \dfrac{4^3}{x^3} & x \geq 4 \end{cases}$$

$$\mathrm{E}[X] = \int_0^\infty S(x)dx = \int_0^4 1\,dx + \int_4^\infty \frac{4^3}{x^3}dx$$

$$= 4 + \frac{-1}{2} \cdot \frac{4^3}{x^2}\bigg|_4^\infty = 4 + \frac{4}{2} = \boxed{6}$$

Exercise 2 Cont.

Alternative approach using densities:

$$S(x) = \begin{cases} 1 & x < 4 \\ \dfrac{4^3}{x^3} & x \geq 4 \end{cases}$$

$$f(x) = \begin{cases} 0 & x < 4 \\ \dfrac{3 \cdot 4^3}{x^4} & x > 4 \end{cases}$$

$$\mathrm{E}[X] = \int_4^\infty x \cdot \frac{3 \cdot 4^3}{x^4}\,dx = \int_4^\infty (3 \cdot 4^3) \cdot x^{-3}\,dx$$

$$= 3 \cdot 4^3 \cdot \frac{x^{-2}}{-2}\bigg|_4^\infty = 3 \cdot 4^3 \cdot \frac{1}{2 \cdot 4^2} = \boxed{6}$$

Or: X is a single parameter Pareto with $\alpha = 3$ and $\theta = 4$, so
$\mathrm{E}[X] = \dfrac{\alpha\theta}{\alpha - 1} = \dfrac{3 \cdot 4}{3 - 1} = \boxed{6}$