A.2 Bayesian Credibility - Outline

A.2.0a Recognizing Distributions

Example 1

Suppose that $f(x) = cx^3$ for $0 < x < 10$. Find $\mathrm{E}[X]$.

$$1 = \int_0^{10} cx^3\,dx = c\int_0^{10} x^3\,dx$$

$$c = \frac{1}{\int_0^{10} x^3\,dx} = \frac{1}{10^4/4}$$

$$c = \frac{4}{10^4}$$

$$f(x) = cx^3 = \frac{x^3}{\int_0^{10} x^3\,dx}$$

$$\mathrm{E}[X] = \int_0^{10} x \cdot f(x)\,dx = \int_0^{10} \frac{4x^4}{10^4} dx$$

$$\mathrm{E}[X] = \frac{4 \cdot 10^5}{5 \cdot 10^4} = \boxed{8}$$

Example 2

Suppose that $f(x) = ce^{-x/3}$ for $0 < x < \infty$. Find Var$[X]$.
Method 1:

- Find $c = \frac{1}{\int_0^\infty e^{-x/3}dx}$
- Find $\mathrm{E}[X] = \int_0^\infty x \cdot ce^{-x/3}dx$
- Find $\mathrm{E}[X^2] = \int_0^\infty x^2 \cdot ce^{-x/3}dx$
- $\mathrm{Var}[X] = \mathrm{E}[X^2] - (\mathrm{E}[X])^2$

Method 2: X is exponential with mean 3, so $\mathrm{Var}[X] = 3^2 = 9$.

Method 2 is faster.

Remark: $c = 1/3$ but we didn't need to know it.

Example 3

Suppose that $f(x) = cx^3e^{-x/3}$ for $0 < x < \infty$. Find Var$[X]$.
Here, just finding $\frac{1}{c} = \int_0^\infty x^3e^{-x/3}dx$ is annoying.

But $f(x)$ is the density of a nice distribution:
If X is a Gamma(α, θ) distribution then
$f(x) = cx^{\alpha-1}e^{-x/\theta}$ for $c = \frac{1}{\theta^\alpha\Gamma(\alpha)}$.

We have an $e^{-x/3}$ term, so $e^{-x/\theta} = e^{-x/3}$ and $\theta = 3$.
We have an x^3 term, so $x^{\alpha-1} = x^3$ and $\alpha = 4$.

$\mathrm{Var}[X] = \alpha\theta^2 = 4 \cdot 3^2 = \boxed{36}$

Remark: We didn't need to know c, but $c = \frac{1}{3^4 \cdot 6}$

Gamma Family

If the density contains an exponential term, and the range is $0 < X < \infty$ it is probably part of the Gamma family.

For the exponential and Gamma distributions, the exponential term is $e^{-\text{variable/something}}$.

For the Weibull distribution, we get $e^{-(\text{variable/something})^{\text{power}}}$

For inverse exponential and Inverse Gamma distributions, it is $e^{-\text{something/variable}}$

In all of those cases, we look at the exponent first, then the rest of the density second.

Gamma Family Exercises 1

Find the mean of the variables with the following densities:

$$
\begin{aligned}
f_X(x) &= \frac{3}{x^2} e^{-3/x} \\
X &\sim \text{Inv. Exp.}(\theta = 3) \\
\mathrm{E}[X] &= \infty \\
f_Y(y) &= 32y^2 e^{-4y} \\
Y &\sim \text{Gamma}(\theta = 1/4, \alpha = 3) \\
\mathrm{E}[Y] &= \alpha\theta = \frac{3}{4} \\
f_X(x) &= \frac{1}{2\sqrt{3x}} e^{-(x/3)^{1/2}} \\
X &\sim \text{Weibull}(\tau = 1/2, \theta = 3) \\
\mathrm{E}[X] &= 3\Gamma\left(1 + \frac{1}{1/2}\right) = 3\Gamma(3) = 3 \cdot (3-1)! = 6
\end{aligned}
$$

Note: we knew, but never used, the value of the constants out front.

Gamma Family Exercises 2

Find the mean of the variables with the following densities:

$$f_Y(y) = \frac{c}{y^3} e^{-1/(2y)}$$

$$Y \sim \text{Inv. Gamma}(\theta = 1/2, \alpha + 1 = 3, \alpha = 2)$$

$$\mathrm{E}[Y] = \frac{1/2}{2-1} = \frac{1}{2}$$

$$f_Z(z) = \frac{c}{\sqrt{z}} e^{-4\sqrt{z}}$$

$$Z \sim \text{Weibull}(\tau = 1/2, 1/\theta^\tau = 4, \theta = 1/16)$$

$$\mathrm{E}[Z] = \frac{\Gamma(3)}{16} = \frac{2!}{16} = \frac{1}{8}$$

$$\pi(\theta) = c\theta^3 e^{-2\theta}$$

$$\theta \sim \text{Gamma}(\text{Scale parameter} = 1/2, \alpha - 1 = 3, \alpha = 4)$$

$$\mathrm{E}[\theta] = 4 \cdot \frac{1}{2} = 2$$

Other Distributions

Without an exponential term, look at the range of possible values.

If the variable ranges from 0 to ∞ it may be a Pareto.
Is the density $\frac{\text{blah}}{(\text{variable} + \text{something})^{\text{power}}}$?

If the range is from c to ∞ it may be a single parameter Pareto.
Is the density $\frac{\text{blah}}{(\text{variable})^{\text{power}}}$?

If the variable ranges from 0 to 1 it may be a Beta.
Is the density $(\text{variable})^{a-1}(1 - \text{variable})^{b-1}$?

Note: Exam tables list the density of a beta as

$$cu^a(1-u)^{b-1} \cdot \frac{1}{x} \qquad u = \frac{x}{\theta}$$

But this simplifies to $cu^{a-1}(1-u)^{b-1} \cdot \frac{1}{\theta}$

More Examples

Find the mean of the variables with the following densities:

$$\pi(\beta) = \frac{32}{(4+\beta)^3} \quad \beta > 0$$
$$\beta \sim \text{Pareto}(\theta = 4, \alpha = 2)$$
$$\text{E}[\beta] = \frac{4}{2-1} = 4$$
$$f(w) = \frac{c}{(2w+5)^4} = \frac{c'}{(w+2.5)^4} \quad w > 0$$
$$W \sim \text{Pareto}(\theta = 2.5, \alpha = 3)$$
$$\text{E}[W] = \frac{2.5}{3-1} = 1.25$$
$$\pi(\theta) = 280\theta^3(1-\theta)^4 \quad 0 < \theta < 1$$
$$\theta \sim \text{Beta}(a = 3+1, b = 4+1)$$
$$\text{E}[\theta] = \frac{4}{4+5} = \frac{4}{9}$$

Mixed Exercises

Find the mean of the variables with the following densities:

$$f(y) = \frac{c}{2y^5} \quad y > 10$$
$$Y \sim \text{SPP}(\theta = 10, \alpha = 4)$$
$$\text{E}[Y] = \frac{\alpha\theta}{\alpha - 1} = \frac{40}{3}$$
$$f(\lambda) = c\lambda^3 e^{-\lambda/10} \quad \lambda > 0$$
$$\Lambda \sim \text{Gamma}(\theta = 10, \alpha = 3+1 = 4)$$
$$\text{E}[\Lambda] = 4 \cdot 10 = 40$$
$$\pi(\lambda) = \frac{\lambda^2\theta^3}{2} e^{-\theta\lambda} \quad \lambda > 0$$
$$\Lambda \sim \text{Gamma}(\text{scale par.} = 1/\theta, \alpha = 3)$$
$$\text{E}[\Lambda] = 3 \cdot \frac{1}{\theta} = \frac{3}{\theta}$$