A.2 Bayesian Credibility - Outline

A.2.4 Poisson/Gamma Conjugate Prior

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- Poisson/Gamma Pairs
- Conjugate Priors
- Exercises

Example

For a specific insured, the number of annual losses are independent Poisson(λ) variables. λ varies by insured, and is Gamma($\alpha = 3, \theta = 1/5$) distributed.
A randomly chosen insured is observed to have 2 losses in 2011, and 4 losses in 2012.
How many losses is that insured expected to have in 2013?

$$
\begin{aligned}
\mathrm{P}[N = k \mid \lambda] &= e^{-\lambda} \cdot \frac{\lambda^k}{k!} \\
\pi(\lambda) &= \frac{\lambda^{3-1}}{\Gamma(3)(1/5)^3} e^{-\lambda/(1/5)} \\
&= c \cdot \lambda^2 e^{-5\lambda} \\
\pi(\lambda \mid N_1 = 2, N_2 = 4) &= c' \cdot \pi(\lambda) \cdot \mathrm{P}[N_1 = 2, N_2 = 4 \mid \lambda] \\
&= c'' \lambda^2 e^{-5\lambda} \cdot e^{-\lambda} \frac{\lambda^2}{2!} \cdot e^{-\lambda} \frac{\lambda^4}{4!} \\
&= c''' \lambda^8 e^{-7\lambda}
\end{aligned}
$$

Example (Continued)

We want $\mathrm{E}[N_3 \mid N_1 = 2, N_2 = 4]$

We found $\pi(\lambda \mid N_1 = 2, N_2 = 4) = c'''\lambda^8 e^{-7\lambda}$

So $(\lambda \mid \text{data}) \sim \text{Gamma}(\alpha' = 9, \theta' = 1/7)$

$$\begin{aligned}
\mathrm{E}[N_3 \mid \text{data}] &= \mathrm{E}[\mathrm{E}[N_3 \mid \lambda] \mid \text{data}] \\
&= \mathrm{E}[\lambda \mid \text{data}] \\
&= \alpha' \cdot \theta' \\
&= \frac{9}{7}
\end{aligned}$$

Poisson/Gamma Pairs

For a specific insured, the number of annual losses are independent Poisson(λ) variables. λ varies by insured, and is Gamma(α, θ) distributed.
A randomly chosen insured is observed to have k losses in n years. What is the posterior distribution of λ?

The # of losses over n years is Poisson $(n\lambda)$

$$\begin{aligned}
\mathrm{P}[\text{data} \mid \lambda] &= c\lambda^k \cdot e^{-n\lambda} \\
\pi(\lambda) &= c'\lambda^{\alpha-1} e^{-\lambda/\theta} \\
\pi(\lambda \mid \text{data}) &\propto \pi(\lambda) \cdot \mathrm{P}[\text{data} \mid \lambda] \\
&= c''\lambda^{\alpha+k-1} e^{-\lambda(n+1/\theta)} \\
(\lambda \mid \text{data}) &\sim \text{Gamma}\left(\alpha' - 1 = \alpha + k - 1, \frac{1}{\theta'} = n + \frac{1}{\theta}\right) \\
(\lambda \mid \text{data}) &\sim \text{Gamma}\left(\alpha' = \alpha + k, \frac{1}{\theta'} = n + \frac{1}{\theta}\right)
\end{aligned}$$

Example (Revisited)

For a specific insured, the number of annual losses are independent Poisson(λ) variables. λ varies by insured, and is Gamma($\alpha = 3, \theta = 1/5$) distributed.
A randomly chosen insured is observed to have 2 losses in 2011, and 4 losses in 2012. How many losses is that insured expected to have in 2013?

We have $k = 6$ losses in $n = 2$ years.
So the posterior is Gamma $\left(\alpha' = 3 + 6, \frac{1}{\theta'} = 2 + \frac{1}{1/5}\right)$
Posterior is Gamma($\alpha' = 9, \theta' = 1/7$)

$$\begin{aligned} \mathrm{E}[N_3 \mid \text{data}] &= \mathrm{E}[\mathrm{E}[N_3 \mid \lambda] \mid \text{data}] \\ &= \mathrm{E}[\lambda \mid \text{data}] \\ &= \alpha' \cdot \theta' = \frac{9}{7} \end{aligned}$$

Conjugate Priors

A Poisson/Gamma pair is an example of a "conjugate prior"

- The prior and posterior have the same type of distribution.
- The posterior predictive distribution is the same type of distribution as the unconditional.

Most continuous Bayesian exam questions involve conjugate priors since they work out nicely.

Some harder questions don't.

Study Strategy:

- At first, do conjugate prior problems both from first principles and using shortcuts
- This will help prepare you for harder problems
- Last 2-4 weeks before exam, pick one approach

Terminology

The textbook and syllabus give reversed names to conjugate priors.
Textbook naming: (prior distribution) - (conditional distribution)
Syllabus naming: (conditional distribution) / (prior distribution)

E.g. syllabus names

- Poisson/Gamma
- Binomial/Beta
- Normal/Normal

Text names

- Gamma - Poisson
- Beta - Binomial
- Normal - Normal

I will use syllabus naming convention. Exam problems will explicitly say (Distribution of (variable) given (parameter) is yadda yadda, distribution of parameter is blah blah).

Exercise 1

For a group health policy, the number of annual claims for each member has a Poisson distribution with mean λ. For each member of a group, λ is the same, but λ varies between groups, having a Gamma distribution with mean 0.06 and variance 0.0002.

For a particular group, 50 claims were observed over the last 4 years. During each of those years, there were 120 members in the group. Find the expected value of λ given those observations.

Exercise 1

For a group health policy, the number of annual claims for each member has a Poisson distribution with mean λ. For each member of a group, λ is the same, but λ varies between groups, having a Gamma distribution with mean 0.06 and variance 0.0002.

For a particular group, 50 claims were observed over the last 4 years. During each of those years, there were 120 members in the group. Find the expected value of λ given those observations.

$$
\begin{aligned}
\mathrm{E}[\lambda] &= \alpha\theta = 0.06 \\
\mathrm{Var}[\lambda] &= \alpha\theta^2 = 0.0002 \\
\theta &= \frac{0.0002}{0.06} = \frac{1}{300} \\
\alpha &= 18
\end{aligned}
$$

Exercise 1 (Continued)

Observe 50 claims in 4 years, with 120 members at risk each year.
Claims in 4 years $\sim$ Poisson $(4 \cdot 120\lambda)$
Prior for λ is Gamma$(\alpha = 18, \theta = 1/300)$

$$
\begin{aligned}
\pi(\lambda \mid \text{data}) &= c \cdot \pi(\lambda) \cdot \mathrm{P}[\text{data} \mid \lambda] \\
&= c' \lambda^{18-1} e^{-\lambda/(1/300)} (480\lambda)^{50} e^{-480\lambda} \\
&= c'' \lambda^{68-1} e^{-780\lambda}
\end{aligned}
$$

$$
\text{Posterior} \sim \text{Gamma}\left(\alpha' = 68, \frac{1}{\theta'} = 780\right)
$$

$$
\mathrm{E}[\lambda \mid \text{data}] = \frac{68}{780}
$$

Using shortcuts, $k = \#\text{claims} = 50,\ n = \#\text{exposures} = 4 \cdot 120,$
$\alpha' = \alpha + k = 18 + 50 = 68,$ and $\dfrac{1}{\theta'} = \dfrac{1}{\theta} + n = 300 + 480 = 780$

Exercise 2

A portfolio consists of a number of independent risks. The number of claims per year for each risk is Poisson distributed with mean λ, where λ varies between risks and has a Gamma($\alpha = 3, \theta = 5$) distribution. During 3 years, 24 claims are observed for a particular risk from this portfolio. What is the variance of the number of claims for this risk in the next year?

Exercise 2

A portfolio consists of a number of independent risks. The number of claims per year for each risk is Poisson distributed with mean λ, where λ varies between risks and has a Gamma($\alpha = 3, \theta = 5$) distribution. During 3 years, 24 claims are observed for a particular risk from this portfolio. What is the variance of the number of claims for this risk in the next year?

First, let's find the posterior of λ.

$$\begin{aligned}
\alpha' &= \alpha + \#\text{claims} = 3 + 24 = 27 \\
1/\theta' &= 1/\theta + \#\text{exposures} = 0.2 + 3 = 3.2 \\
(\lambda \mid \text{data}) &\sim \text{Gamma}(\alpha' = 27, \theta' = 1/3.2) \\
\text{Or: } \pi(\lambda \mid \text{data}) &\propto \pi(\lambda) \cdot \mathrm{P}[\text{data} \mid \lambda] \\
&= c \cdot \lambda^2 e^{-\lambda/5} \cdot (3\lambda)^{24} e^{-3\lambda} \\
&= c' \lambda^{27-1} e^{-3.2\lambda} \\
(\lambda \mid \text{data}) &\sim \text{Gamma}(\alpha' = 27, \theta' = 1/3.2)
\end{aligned}$$

Exercise 2 (Continued)

Key point: We want the variance of the number of claims next year, not the variance of λ.
We want $\text{Var}[N_4 \mid \text{data}]$

$$\begin{aligned}
N &\sim \text{Poisson}(\lambda) \\
(\lambda \mid \text{data}) &\sim \text{Gamma}(\alpha' = 27, \theta' = 1/3.2) \\
(N_4 \mid \text{data}) &\sim \text{Neg. Bin.}(r' = 27, \beta' = 1/3.2) \\
\text{Var}[N_4 \mid \text{data}] &= r'\beta'(1+\beta') = \boxed{11.07} \\
\text{Or: } \text{E}[N_4 \mid \text{data}] &= \text{E}[\text{E}[N_4 \mid \lambda] \mid \text{data}]] \\
&= \text{E}[\lambda \mid \text{data}] = \alpha'\theta' = 8.44 \\
\text{Var}[N_4 \mid \text{data}] &= \text{E}[\text{Var}[N_4 \mid \lambda] \mid \text{data}]] + \text{Var}[\text{E}[N_4 \mid \lambda] \mid \text{data}]] \\
&= \text{E}[\lambda \mid \text{data}] + \text{Var}[\lambda \mid \text{data}] \\
&= \alpha'\theta' + \alpha'\theta'^2 = \boxed{11.07}
\end{aligned}$$