A.3 Bühlmann Credibility - Outline

Bühlmann Credibility

An insurance company covers two types of insureds:

- high risk insureds have a mean loss of 2,000 and SD of 1
- low risk insureds have a mean loss of 1,000 and SD of 1.

Qualitatively speaking, how many claims does it take to determine whether or not an insured is high risk?

1. If the first data point is near 2,000 (e.g., 1,998 or 2,001) then the person is, with extreme certainty, high risk;
If the first data point is 1,001 or something else near 1,000, then they are most likely low risk.

In Bühlmann language, the VHM (Variance of the Hypothetical Means) is huge compared to the EPV (Expected Process Variance).

Bühlmann Credibility

An insurance company has two types of insureds:

- high risk insureds have a mean loss of 1,002 and SD of 500
- low risk insureds have a mean loss of 1,000 and SD of 500.

Qualitatively speaking, how many claims does it take to determine whether or not an insured is high risk?

Lots, since the data values will be all over the place from 500 to 1,500 for both groups.

In Bühlmann language, the VHM (Variance of the Hypothetical Means) is small compared to the EPV (Expected Process Variance).

Bühlmann Credibility

Notation:

$$X \sim \text{Distribution}(\theta), \text{ where } \theta \text{ varies between policyholders}$$

$$\mathrm{E}[X \mid \theta] = \mu_X(\theta) \qquad \mathrm{Var}[X \mid \theta] = \sigma_X^2(\theta)$$

$$\mu_X = \mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X \mid \theta]] = \mathrm{E}[\mu_X(\theta)]$$

$$\mu_{PV} = \text{Expected Process Variance (EPV)} = \mathrm{E}[\mathrm{Var}[X \mid \theta]] = \mathrm{E}[\sigma_X^2(\theta)]$$

$$\sigma_{HM}^2 = \text{Variance of the Hypothetical Means (VHM)} = \mathrm{Var}[\mathrm{E}[X \mid \theta]]$$

$$k = \frac{\mu_{PV}}{\sigma_{HM}^2} = \frac{\text{EPV}}{\text{VHM}} \qquad \text{“Bühlmann's k”}$$

Suppose we have data $X_1, \ldots, X_n$ with sample mean $\overline{X}$.

$$P_C = \mu_X(1 - Z) + Z \cdot \overline{X} = \mu_X + Z(\overline{X} - \mu_X)$$

$$Z = \frac{n}{n + \mu_{PV}/\sigma_{HM}^2} = \frac{n}{n+k} \qquad \text{“Bühlmann credibility factor”}$$

Intuition: $k =$ # data points needed for $Z = 1/2$

Bühlmann Credibility

An insurance company has two types of insureds:

- high risk insureds have a mean loss of 2,000 and SD of 1
- low risk insureds have a mean loss of 1,000 and SD of 1.

If half of the insureds are high risk, calculate Bühlmann's k.

$$\mu_{PV} = \mathrm{E}[\mathrm{Var}[X \mid \text{Risk type}]] = 1^2$$
$$\sigma^2_{HM} = \mathrm{Var}[\mathrm{E}[X \mid \text{Risk type}]] = (2{,}000 - 1{,}000)^2 \cdot 0.5^2$$
$$k = \frac{1}{250,000}$$

Note: $\mathrm{Var}[X] = \mu_{PV} + \sigma^2_{HM} = 250{,}001$

Bühlmann Credibility

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If half of the insureds are high risk, calculate Bühlmann's k.

$$\mu_{PV} = \mathrm{E}[\mathrm{Var}[X \mid \text{Risk type}]] = 500^2 = 250{,}000$$
$$\sigma^2_{HM} = \mathrm{Var}[\mathrm{E}[X \mid \text{Risk type}]] = 2^2 \cdot (0.5)^2 = 1$$
$$k = \frac{\mu_{PV}}{\sigma^2_{HM}} = 250{,}000$$

Note: $\mathrm{Var}[X] = \mu_{PV} + \sigma^2_{HM} = 250{,}001$
We have the same total variance in both examples just a different split.

Historical note: Previous exams and texts use $v = \mu_{PV}$ and $a = \sigma^2_{HM}$

Example

Annual losses are Pareto with mean $\theta/2$ and variance $3\theta^2/4$, where θ varies by insured and has a Gamma distribution with mean 10 and variance 20.
A randomly selected insured has losses totaling 30 in 3 years. Find the Bühlmann credibility premium for this insured next year.

$$\mathrm{E}[X \mid \theta] = \frac{\theta}{3-1} = \frac{\theta}{2} \qquad \mathrm{Var}[X \mid \theta] = \left(\frac{\theta^2}{4}\right)\frac{3}{3-2} = \frac{3\theta^2}{4}$$

$$\mu_X = \mathrm{E}[X] = \mathrm{E}[\mathrm{E}[X \mid \theta]] = \mathrm{E}\left[\frac{\theta}{2}\right] = 5$$

$$\mu_{PV} = \mathrm{E}[\mathrm{Var}[X \mid \theta]] = \mathrm{E}\left[\frac{3\theta^2}{4}\right] = \frac{3}{4}(20 + 10^2) = 90$$

$$\sigma^2_{HM} = \mathrm{Var}[\mathrm{E}[X \mid \theta]] = \mathrm{Var}\left[\frac{\theta}{2}\right] = \frac{\mathrm{Var}[\theta]}{4} = 5$$

Example (Cont)

Annual losses are Pareto with mean $\theta/2$ and variance $3\theta^2/4$, where θ varies by insured and has a Gamma distribution with mean 10 and variance 20.
A randomly selected insured has losses totaling 30 in 3 years. Find the Bühlmann credibility premium for this insured next year.

$$\begin{aligned} \mu_X &= 5 \\ \mu_{PV} &= 90 \\ \sigma^2_{HM} &= 5 \\ P_C &= \mu_X + \frac{n}{n + \mu_{PV}/\sigma^2_{HM}}\left(\overline{X} - \mu_X\right) \\ &= 5 + \frac{3}{3 + 90/5}\left(\frac{30}{3} - 5\right) = \boxed{5.7} \end{aligned}$$

Exercise 1

Individual losses are exponential(θ), where θ varies by insured, and is uniform on $(0, 10)$. Six losses from a randomly chosen insured are observed, with values 3, 19, 12, 8, 32, and 16. Find the Bühlmann credibility premium for the next loss for this insured.

Exercise 1

Individual losses are exponential(θ), where θ varies by insured, and is uniform on $(0, 10)$. Six losses from a randomly chosen insured are observed, with values 3, 19, 12, 8, 32, and 16. Find the Bühlmann credibility premium for the next loss for this insured.

$$\overline{X} = \frac{3 + 19 + 12 + 8 + 32 + 16}{6} = 15$$

$$\mathrm{E}[X \mid \theta] = \theta \qquad \mathrm{Var}[X \mid \theta] = \theta^2$$

$$\mu_X = \mathrm{E}[\mathrm{E}[X \mid \theta]] = \mathrm{E}[\theta] = 5$$

$$\mu_{PV} = \mathrm{E}[\mathrm{Var}[X \mid \theta]] = \mathrm{E}[\theta^2]$$

$$= \mathrm{Var}[\theta] + (\mathrm{E}[\theta])^2 = \frac{10^2}{12} + 5^2 = \frac{100}{3}$$

$$\sigma^2_{HM} = \mathrm{Var}[\mathrm{E}[X \mid \theta]] = \mathrm{Var}[\theta] = \frac{10^2}{12} = \frac{100}{12}$$

$$P_C = 5 + \frac{6}{6 + (100/3)/(100/12)}(15 - 5) = \boxed{11}$$

Note: $\theta < 10$, so $\mathrm{E}[X \mid \theta] < 10$ and 11 is too high to be a realistic estimate. This is a problem with Bühlmann credibility.

Exercise 2

Monthly losses are Gamma($\alpha = 2, \theta$), where θ varies by insured, and has density $\pi(\theta) = 5 \cdot 12^5 \cdot (\theta + 12)^{-6}$

In the last four months, a randomly chosen insured had losses of 6, 12, 15, and 7. Find the Bühlmann credibility premium for the next 3 months of losses for this insured.

Exercise 2

Monthly losses are Gamma($\alpha = 2, \theta$), where θ varies by insured, and has density $\pi(\theta) = 5 \cdot 12^5 \cdot (\theta + 12)^{-6}$

In the last four months, a randomly chosen insured had losses of 6, 12, 15, and 7. Find the Bühlmann credibility premium for the next 3 months of losses for this insured.

$$\mathrm{E}[X \mid \theta] = 2\theta \qquad \mathrm{Var}[X \mid \theta] = 2\theta^2$$

$$\theta \sim \text{Pareto}(\alpha = 5, \text{scale param} = 12)$$

$$\mathrm{E}[\theta] = \frac{12}{5-1} = 3 \qquad \mathrm{E}[\theta^2] = \frac{2 \cdot 12^2}{4 \cdot 3} = 24$$

$$\mu_X = \mathrm{E}[\mathrm{E}[X \mid \theta]] = \mathrm{E}[2\theta] = 6$$

$$\mu_{PV} = \mathrm{E}[\mathrm{Var}[X \mid \theta]] = \mathrm{E}[2\theta^2] = 48$$

$$\sigma^2_{HM} = \mathrm{Var}[\mathrm{E}[X \mid \theta]] = \mathrm{Var}[2\theta] = 2^2 \cdot (24 - 3^2) = 60$$

$$\overline{X} = \frac{6 + 12 + 15 + 7}{4} = 10$$

$$3P_C = 3\left[6 + \frac{4}{4 + 48/60}(10 - 6)\right] = \boxed{28}$$