THE INFINITE ACTUARY'S

SAMPLE DETAILED STUDY MANUAL FOR THE

INV-201 Exam

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Hull Ch 19: The Greek Letters

John Hull (2021)

Overview of This Reading

This chapter explores the idea of a financial institution hedging the risk from selling a call option to a client. First, simple hedging choices such as using covered and stop-loss positions are discussed. Next, hedging using the "Greeks" is discussed.

Key topics for the exam include:

- Describe covered and stop-loss positions
- Describe and calculate Greeks under the Black-Scholes framework

Overview

- A financial institution that sells an option in over-the-counter markets is faced with the problem of managing its risk
  - If the option happens to be the same as one that is traded actively, the financial institution can neutralize its exposure by buying the same option
  - But when the option has been tailored to the needs of a client and does not correspond to the products traded on exchanges, hedging is far more difficult
- Thus, financial institution selling options faces risk management challenges
- This chapter covers risk management through "Greeks"
- Each Greek letter measures a different dimension of risk
- A trader may aim to manage the Greeks so that risks are acceptable

19.1. Illustration

Suppose a financial institution has sold a European call option on 100,000 shares of a non-dividend-paying stock for $300,000.

The given parameters[1] are:

- $S_0 = 49$
- $K = 50$
- $r = 5\%$

[1]Note that the expected return $\mu = 13\%$ is irrelevant to the pricing of an option.

- $\mu = 13\%$
- $\sigma = 20\%$
- $T = .3846$

The Black-Scholes price[2] is about $240,000; the company made a "theoretical" profit of $60,000. However, the financial institution is left with the problem of hedging their risks. Different ways their risk could be managed are discussed next.

19.2. Naked and Covered Positions

Naked and Covered Positions

- In a naked position, a financial institution does nothing to hedge the risk of a short option; this will work well if the option is not exercised
- In a covered[3] position, the financial institution takes a position in the underlying security
  - In the chapter's example, the company would purchase 100,000 shares of stock
  - This would protect against a rise in the stock price, but would be exposed to a drop in the price
- Neither a naked position nor a covered position provides a good hedge

Stop-Loss Strategy

- To cover a short call, the underlying stock is purchased when the option is in-the-money (above the strike price) and sold when it is out-of-the-money (below the strike price)
- This is a blend of the naked and covered positions
- The procedure is designed so that at time T the institution owns the stock if the option closes in-the-money and does not own it if the option expires out-of-the-money
- This strategy could work well if there were no transaction costs (which is clearly not true)
- Purchases are made at a price of $K + \epsilon$ and sales are made at a price of $K - \epsilon$
  - This has an underlying cost, because this involves "buying high" and "selling low"
  - Every purchase and subsequent sale has a cost of 2ϵ in addition to transaction costs

[2] Note in the TIA spreadsheet for this lesson, we derive the Black-Scholes call price (per share) based on these parameters is $2.40.

[3] For example, a covered call is created by holding the underlying security and selling a call option.

- Might hypothesize that the total cost of writing and hedging the option Q using the stop-loss strategy is equal to the option's intrinsic value $Q = \max(S_0 - K, 0)$. However, this equation is incorrect:
  - Cash flows to the hedger occur at different times and must be discounted
  - Purchases and sales are made at different prices
  - In other words, if the hedging strategy is "buying high" and "selling low", this is an additional cost. And yet another cost is the transaction costs from hedging. Thus, the cost of writing and hedging can exceed the option's intrinsic value due to these costs
- ϵ can be reduced by trading more frequently, but then the number of trades increase
- Stop-loss hedging strategy does not work particularly well
  - Can measure hedge performance as the ratio of the standard deviation of the cost of hedging the option to the Black–Scholes option price
  - Effective hedging scheme should have a hedge performance measure close to zero
  - Simulation results show a ratio of .76-.98, far above 0

19.3. Greek Letter Calculation

- Most traders use more sophisticated hedging procedures than those mentioned so far
- These hedging procedures involve calculating measures such as delta, gamma, and vega
- In order to calculate a Greek letter, it is necessary to assume an option pricing model
- Traders often assume the Black-Scholes model
- The "practitioner" Black-Scholes model uses the current implied volatility as the volatility value when calculating Greeks

19.4. Delta Hedging

- The **delta** is the rate of change of the option price with respect to the price of the underlying asset
- For a call option, $\Delta = \frac{\partial C}{\partial S}$
- Example: Suppose that the delta of a call option on a stock is 0.6. This means that when the stock price changes by a small amount, the option price changes by about 60% of that amount
- A short call option could be hedged by purchasing delta times the number of shares in the call option
- A position with a delta of zero is referred to as being *delta neutral*

- Delta changes continuously and does not remain constant
- Dynamic hedging adjusts on a regular basis, whereas a static hedge is set up initially and never adjusted

Delta of European Stock Options

- Using the Black-Scholes formula on a non-dividend paying stock:
  - $\Delta_{\text{Call}} = N(d_1)$
  - $\Delta_{\text{Put}} = N(d_1) - 1$
- As the stock price increases, the delta for a call (put) approaches 1.0 (0.0)
- As the stock price decreases, the delta for a call (put) approaches 0.0 (-1.0)

Dynamic Aspects of Delta Hedging

- Delta will approach one for a call option as the maturity date nears if the option is in-the-money
- The delta will approach zero as the option maturity date nears if the option is out-of-the-money
- If the hedging scheme worked perfectly, the present value of the cost would equal the Black-Scholes value of the option
  - This assumes constant volatility and no transaction costs
- Delta hedging works much better than the stop-loss strategy

Where the Cost Comes From

- Delta hedging synthetically creates a position in the underlying security to hedge the option
- In hedging a short call, the synthetic position is a long security
- Since the delta for a call increases as the underlying security rises, delta hedging requires buying high and selling low (not exactly a good trading strategy)
- This creates the cost for delta hedging

Delta of a Portfolio

- The delta of a portfolio Π is:

$$\frac{\partial \Pi}{\partial S} = \sum_{i=1}^{n} w_i \Delta_i$$

- Here, w_i is the number of options for the ith security
- Δ_i is the delta of the ith security (per option)
- A position in the underlying security or futures/forward contract can be used to make a portfolio delta neutral

Transaction Costs

- Delta-neutral positions in a single option and underlying asset can be cost prohibitive, especially for smaller portfolios
- Usually rebalance once per day to maintain delta neutrality

19.5. Theta

- The **theta** (i.e. time decay) of an option is the rate of change of the option with respect to the passage of time
- The Black-Scholes formulas for theta are:

$$\Theta(\text{call}) = -\frac{S_0 N'(d_1)\,\sigma}{2\sqrt{T}} - rKe^{-rT} N(d_2)$$
$$\Theta(\text{put}) = -\frac{S_0 N'(d_1)\,\sigma}{2\sqrt{T}} + rKe^{-rT} N(-d_2)$$

- Since $N(-d_2) = 1 - N(d_2)$, theta of a put exceeds theta of a call by rKe^{-rT}
- Where we have that:
  - N is the standard normal CDF
  - N′ is the standard normal PDF
  - $N'(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}}$

Tip: Standard Normal Functions in Excel

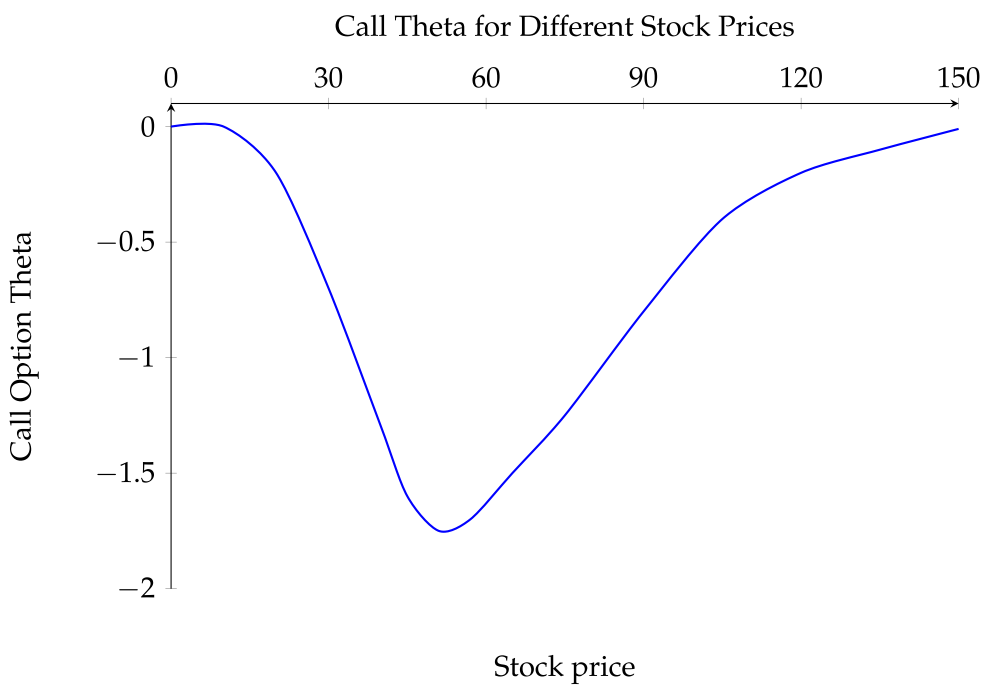
Note that values from the standard normal PDF N' and the standard normal CDF N can be quickly computed using Excel functions. Since Excel is available at Prometric, **you should be familiar with the functions listed below.** Check out the TIA video series spreadsheet example for further practice!

PDF: $N'(d_1) \rightarrow$ NORM.S.DIST(d_1,0)

CDF: $N(d_1) \rightarrow$ NORM.S.DIST(d_1,1)

The second input to the function NORM.S.DIST is used to specify whether you would like to use the standard normal PDF (0 or false) or CDF (1 or true).

- Formulas above measure theta in terms of years
  - Can divide by 365 to scale in terms of theta per calendar day, or 252 to scale in terms of theta per trading day[4]
- There is no uncertainty about the passage of time, so it does not make sense to hedge using theta; it is a useful descriptive statistic
- Theta is usually negative for a long option because option values decrease as there is less time to maturity
- Call option interpretation:
  - When the stock price is very low, call theta is close to zero
  - For an at-the-money call option, theta is large in magnitude and negative
  - As the stock price becomes larger, theta tends to $-rKe^{-rT}$ (the graph below uses $r = 0$ and $K = 50$)
- Theta is a proxy for gamma in a delta-neutral portfolio



[4] This assumes there are 252 trading days per year and 365 calendar days per year.

19.6. Gamma

- The **gamma** is the rate of change of delta with respect to the underlying security; this is the second partial derivative of the portfolio with respect to the asset price
- $\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$
- If gamma is small, delta changes slowly, and adjustments to keep a portfolio delta neutral need to be made only relatively infrequently
- If gamma is very positive or very negative then delta will change rapidly
- Gamma measures the curvature in the option price / stock price graph
- For a delta-neutral portfolio, the change in the portfolio value can be approximated with:

$$\Delta\Pi = \Theta\Delta t + \frac{1}{2}\Gamma\Delta S^2$$

- When gamma is positive, theta tends to be negative; when gamma is negative, theta tends to be positive
- Delta neutrality provides protection against relatively small stock price moves between rebalancing, and gamma neutrality provides protection against larger movements in stock price between hedge rebalancing

Making a Portfolio Gamma Neutral

- Positions in the underlying asset or forward contracts on the asset have zero gamma, so they cannot be used to alter the gamma of a portfolio
- A security like a traded option is needed to alter the gamma
- Suppose that a delta-neutral portfolio has a gamma equal to Γ
- Suppose there is a traded option that has a gamma equal to Γ_T
- If the number of traded options added to the portfolio is w_T, the gamma of the portfolio is:

$$w_T\Gamma_T + \Gamma$$

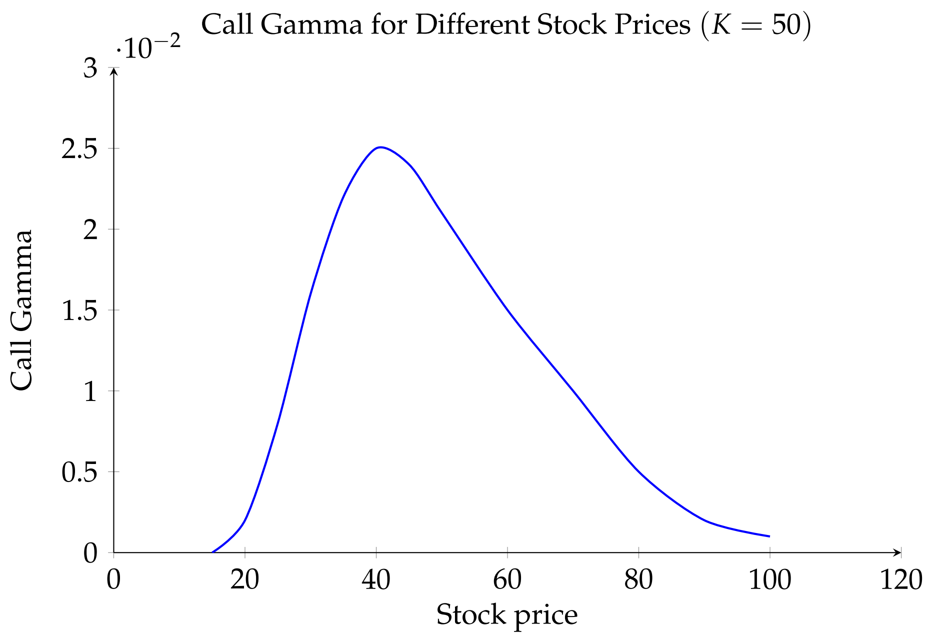
- Thus, the position in the traded option necessary to make the portfolio gamma neutral is:

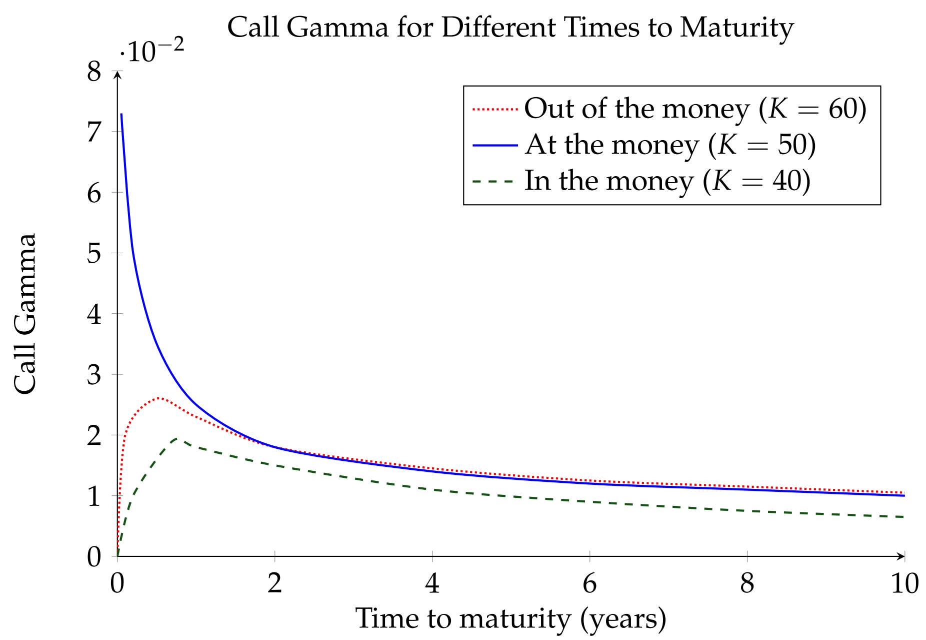
$$w_T\Gamma_T + \Gamma = 0 \Rightarrow w_T = -\frac{\Gamma}{\Gamma_T}$$

- Must balance gamma before delta, because the securities used to adjust the gamma will also adjust delta
- Including the traded option is likely to change the delta of the portfolio, so the position in the underlying asset then has to be changed to maintain delta neutrality

Calculation of Gamma

- $\Gamma = \frac{N'(d_1)}{S_0 \cdot \sigma \sqrt{T}}$
- For an at-the-money option, gamma increases as time to maturity decreases
- Short-life at-the-money options have very high gammas
- The following graphs summarize gamma for a European call option as the stock price and time to maturity vary





Gamma Neutral Example

Suppose that a portfolio is delta neutral and has a gamma of $-6{,}000$.

A particular traded call option has a delta of 0.60 and gamma of 1.50.

Describe how to make the portfolio delta and gamma neutral.

Solution:

Purchase $w_T = -\frac{\Gamma}{\Gamma_T} = -\frac{-6{,}000}{1.5} = 4{,}000$ of the traded call option to neutralize gamma.
This purchase increases delta by $.60 \times 4{,}000 = 2{,}400$. Therefore, short 2,400 of the underlying stock to maintain delta neutrality.

19.7. Relationship between Delta, Theta, and Gamma

For a portfolio of non-dividend-paying stocks,

$$\frac{\partial \Pi}{\partial t} + rS\frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi \Rightarrow \boxed{\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi}$$

Note that the above equation is the Black-Scholes PDE.

If the portfolio is delta-neutral ($\Delta = 0$), then this simplifies to:

$$\Theta = r\Pi - \frac{1}{2}\sigma^2 S^2 \Gamma$$

19.8. Vega

- The **vega** of a portfolio is the rate of change in the portfolio with respect to the volatility of the underlying asset
- $v = \frac{\partial \Pi}{\partial \sigma}$
- Note that vega is the name given to one of the "Greek letters" in option pricing, but it is not one of the letters in the Greek alphabet
- Shares in the underlying asset have zero vega, so the stock cannot be used to alter portfolio vega
- A security like a traded option is needed to alter the vega
- Two options would be needed to make the portfolio gamma neutral and vega neutral
- For a European call or put option, $v = S_0\sqrt{T}N'(d_1)$
- The vega in a long position in a regular European or American option is always positive
- It is a bit odd to use the Black-Scholes model to calculate vega since that model assumes a constant volatility; however, it does produce reasonable results

Portfolio Hedging Example

Consider a portfolio that is delta neutral, with a gamma of -5,000 and a vega (measuring sensitivity to implied volatility) of -8,000. The options shown in the following table can be traded:

	Delta	Gamma	Vega
Portfolio	0	-5,000	-8,000
Option A	.6	.5	2.0
Option B	.5	.8	1.2

Using options and a position in the underlying, determine what positions can be added to the portfolio to make it (simultaneously) delta, gamma, and vega neutral.

Solution:

Let A be the number of units of option A, and B be the number of units of option B. Then,

$.5A + .8B = 5{,}000$ and $2A + 1.2B = 8{,}000$

Solving this system of equations gives $A = 400$ and $B = 6{,}000$

However, introducing these options contracts gives delta risk that needs to be removed. We have a delta of $.6A + .5B = 3{,}240$

Thus, the answer is:

- Buy 400 units of option A
- Buy 6,000 units of option B
- Short 3,240 units of the underlying asset

19.9. Rho

- The **rho** of a portfolio of options is the rate of change of the portfolio with respect to the interest rate
- $\rho = \frac{\partial \Pi}{\partial r}$
- r is usually set equal to the risk-free rate for a maturity equal to the option's maturity
- Using the Black-Scholes model for European options:
- $\rho_{\text{Call}} = KTe^{-rT}N(d_2)$
- $\rho_{\text{Put}} = -KTe^{-rT}N(-d_2)$

Greek Summary

The table below summarizes the Black-Scholes Greek formulas together. The reading assumes no dividends. Note that for Gamma and Vega, the call and put formulas are the same.

Greek	Sensitivity	Call Value	Put Value
Delta $\Delta = \frac{\partial \Pi}{\partial S}$	Underlying	$N(d_1)$	$N(d_1) - 1$
Theta $\Theta = \frac{\partial \Pi}{\partial t}$	Time	$-\dfrac{S_0 N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$	$-\dfrac{S_0 N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$
Gamma $\Gamma = \frac{\partial^2 \Pi}{\partial S^2}$	Convexity	$\frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$	
Vega $v = \frac{\partial \Pi}{\partial \sigma}$	Volatility	$S_0 \sqrt{T} N'(d_1)$	
Rho $\rho = \frac{\partial \Pi}{\partial r}$	Interest Rates	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Where we have that:

- N is the standard normal CDF, N′ is the standard normal PDF
- $N'(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{\frac{-x^2}{2}}$
- $d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$, $d_2 = d_1 - \sigma\sqrt{T}$

INV201-109: Investment Instruments with Volatility Targets

Albeverio et. al. (May 2013)

Overview of This Reading

The full title of this study note is "INV201-109: Investment Instruments with Volatility Target Mechanism". This reading explores volatility target funds by looking at the results of numerical simulations and analyzing their results.

Key topics for the exam include:

- Describe the five steps of the VolTarget mechanism
- Identify scenarios where VolTarget performs well compared to pure equity
- Identify scenarios where VolTarget performs poorly compared to pure equity
- Compute the guarantee payoff given initial investment, participation rate, and equity return
- Explain how to compute the risk budget
- State the no-arbitrage participation rate is equal to the risk budget divided by the option cost
- Understand Asian options are cheaper than European options
- State the three drawbacks of the VolTarget mechanism
- Be able to look at the tables seen throughout the detailed study guide and explain the directional movements. While the exact values are not important, it's important to be able to explain in your own words how changing a volatility parameter or the trading strategy (e.g. VolTarget vs pure equity) directionally impacts option prices or participation rates

Section 1: Introduction

- Dramatic market downturns often happen in scenarios with extreme market volatility
- Practitioners often recognize rising volatility as a good indicator for a falling market
- Purchasing put option protection is an expensive choice when volatilies are already high
- Using a volatility target ("VolTarget") mechanism can allow for equity participation with protection from significant losses due to market crashes
- The volatility target portfolio has two components:
  1. Equities
  2. Risk-free bond

- Asset positions are rebalanced so that the overall portfolio volatility level is kept under control
  - When markets become more volatile, weight in risk-free assets is increased
  - When markets become less volatile, weight in equities is increased
  - The sum of the two weights adds up to 100%
- **Cash-in risk:** in extreme market scenarios, when the value of a risky asset decreases significantly, the whole portfolio needs to be invested into a risk-free asset
  - VolTarget approach does not have cash-in risk

Section 2: Construction of a VolTarget Portfolio

VolTarget Description

- Suppose M_{t_0} is invested at time t_0
- This reading assumes the initial investment is all in equities, and then rebalancing is done at t_1 between equities and the risk-free asset using the five steps described below
- Note that Steps 1-2 are typically only done once at time t_0, while Steps 3-5 are completed every time the portfolio is rebalanced

Five Steps of the VolTarget Mechanism

1. **Choose Time Frequencies**
   - Select the length of time, in years, h between portfolio rebalancing (e.g. $h = \frac{1}{12}$ denotes monthly rebalacing)
   - Select the risky asset historical volatility estimate period H
2. **Set the Volatility Target $\bar{V}$**
   - Typically calibrated as the historical volatility of the risky asset over the most recent time period of length H
3. **Estimate V_{t_i}**
   - V_{t_i} is calculated as the historical annualized volatility of the risky asset over the most recent time period of length H
4. **Assign Weights**
   - The portfolio contains two securities: equities and risk-free bonds. The weights of these two securities will add up to 100%
   - To compute the equity weight, take the ratio of the volatility target to the estimated volatility. In order to limit extreme leverage, the equity weight is capped (e.g. never to exceed 200%)

- Equity Weight = $\alpha^1_{t_1} = \min\{\frac{\bar{V}}{V_{t_1}}, 2\}$ and Bond Weight = $\alpha^2_{t_1} = 1 - \alpha^1_{t_1}$
- A superscript of 1 indicates equity, while a superscript of 2 indicates bonds
- Weights are static between rebalancing, and updated every h years
- If (at a rebalancing time) the historical volatility of the risky asset is higher than the chosen volatility target, then the equity weight is between 0 and 100%
- If (at a rebalancing time) the historical volatility of the risky asset is less than the chosen volatility target, then the equity weight is between 100 and 200%. This corresponds to a leveraged equity position with borrowing at the risk-free rate
- *Note: Compare the $\alpha^1_{t_1}$ formula above to the target volatility equity ratio calculation in Appendix B for INV201-106. They are very similar formulas, just using different caps. The 200% cap for the equity weight is one potential value for the cap. The prior reading uses 110% as a cap. On the exam, it's likely you would be given the cap to use if it was needed for calculations.*

5. **Rebalance**
    - Use the portfolio weights from the prior step to rebalance the portfolio
    - For example, if the total value of the portfolio is M_{t_1} at time t_1, then:
        - Equity dollar position is $\alpha^1_{t_1} \times M_{t_1}$ and risk-free bond position is $\alpha^2_{t_1} \times M_{t_1}$

Simulation Results and Remarks

- Over the simulation period, the authors compare a pure-equity vs VolTarget strategy:
    - Equity: 6.95% mean return, 13.56% volatility
    - VolTarget: 6.1% mean return, 13.1% volatility
    - The difference in simulated risk/return profile highly depends on the sample period, but VolTarget typically gives a significant reduction in portfolio volatility compared to equities (especially when the volatility target $\bar{V}$ is calibrated to a sufficiently low value)
- VolTarget performs <u>better</u> (compared to equity only) in scenarios where:
    - A falling market is accompanied by high volatility levels
    - A rising market is accompanied by low volatility levels
- VolTarget performs <u>worse</u> (compared to equity only) in scenarios where:
    - A falling market is accompanied by low volatility levels
    - A rising market is accompanied by high volatility levels
- Determining the appropriate rebalancing frequency h is challenging
    - Monthly is commonly used by practitioners who are looking to minimize transaction costs associated with frequent portfolio rebalancing
    - A monthly rebalanced VolTarget portfolio may be exposed to significant losses in the case of short-term market shocks
    - Need to find the appropriate balance between short-term market risks and rebalancing costs

- Determining H is also challenging
  - Can use monthly estimates of historical volatility over a rolling window of one year
- Other risk measures (e.g. implied volatility) may provide better protection against short-term volatility spikes in the future
  - Historical volatility is backwards looking, and may react more slowly to short-term spikes compared to forward looking measures such as implied volatility

Volatility Target Example

Suppose you are given the following information:

- Initially, at time t_0, an investor creates a volatility target portfolio with a value of 100,000
- At time $t_1 > t_0$, the investment grows to 110,000 and the annualized historical volatility of the stock index is $V_{t_1} = 16\%$.
- At time $t_2 > t_1$, the investment further grows to 120,000 and the annualized historical volatility of the stock index is $V_{t_2} = 10\%$.
- Portfolio rebalancing is done to maintain a volatility target of 12% (i.e. $\bar{V} = 12\%$). Rebalancing is done at both times t_1 and t_2. The securities in the portfolio are equities and risk-free bonds, and the equity position weight is constrained to never exceed 200%

Given the information above, answer the questions below:

(a) What is the risk-free bond and equity position of the volatility target portfolio after rebalancing at time t_1?

(b) What is the risk-free bond and equity position of the volatility target portfolio after rebalancing at time t_2?

(c) Briefly compare the portfolios at times t_1 and t_2

Solution:

Part (a): Rebalancing at time t_1

- Equity Weight = $\alpha^1_{t_1} = \min\{\frac{\bar{V}}{V_{t_1}}, 2\} = \min\{\frac{.12}{.16}, 2\} = .75$ and Bond Weight = $= .25$
- Invest 75% in equities and 25% in bonds, which gives dollar positions of $75\% \cdot 110{,}000 = 82{,}500$ in equities and $25\% \cdot 110{,}000 = 27{,}500$

Part (b): Rebalancing at time t_2

- Equity Weight = $\alpha^1_{t_2} = \min\{\frac{\bar{V}}{V_{t_1}}, 2\} = \min\{\frac{.12}{.10}, 2\} = 1.2$ and Bond Weight = $-.2$
- Invest 120% in equities and -20% in bonds, which gives dollar positions of $120\% \cdot 120{,}000 = 144{,}000$ in equities and $-20\% \cdot 120{,}000 = -24{,}000$ (i.e. borrow money for a leveraged equity position)

Part (c)

- At time t_2, equity markets are strong and historical volatility is low. Therefore, the volatility target portfolio puts a higher weight in equities for t_2 compared to t_1 (lower historical volatility gives a higher equity weight)
- In fact, the t_2 portfolio borrows at the risk-free rate to obtain a leveraged equity position

Section 3: VolTarget Portfolio Linked Derivatives

Theoretical Pricing Background

- The no-arbitrage price V_0 of a contingent claim at time 0 is given by:

$$V_0 = \frac{1}{B_T} E^{\mathbb{P}^*}[f(S)]$$

- Notation
  - $B_T = B_0 e^{rT}$ where there is a constant positive interest rate r
  - $\mathbb{P}^*$ is the risk-neutral measure
  - $f(S)$ is a contingent payoff at time T (e.g. put/call option payoff)
  - *Note: The reading uses big-O notation $\mathcal{O}$ to denote value, I will use V but understand either notation means the same thing. That is, V_0 and $\mathcal{O}_0$ both refer to the option value*

- Discounted stock price process $\bar{S}_t = \frac{S_t}{B_t}$ is a martingale with respect to risk-neutral measure

- We can think of the option as a dynamic combination of stocks and bonds such that:

$$V_t = \beta_k S_t + \gamma_k B_t$$

- The coefficients only change every h timesteps when rebalancing occurs

- Normalizing by the bond price gives that:

$$\bar{V}_t = \beta_k \bar{S}_t + \gamma_k$$

- This tells us that we can think of the VolTarget portfolio as a dynamic combination of stocks and bonds. Asset V is measured in units of the VolTarget portfolio linked to S

- $\bar{V}_t$ is a martingale with respect to the risk-neutral measure

- The author uses a geometric brownian motion assumption for the risky asset stock price:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

European Call Option Results Linked to VolTarget Portfolios

- Using this framework, ATM call option prices can be computed using different risky asset annual volatilies σ:

Underlying Asset	$\sigma = 10\%$	$\sigma = 50\%$	Change
Equity	23.42	49.60	111%
VolTarget ($\bar{V} = 10\%$)	22.35	23.21	4%
VolTarget ($\bar{V} = 20\%$)	28.37	31.19	9%

- Key interpretations from the table include:
  - For the equity row (and all others too), we see the option value increase as volatility increases. This is as we would expect, since options have a positive vega and option values increase when volatilities increase
  - Option prices in the case of the VolTarget underlying demonstrate significantly lower dependence on the risky asset volatility compared to the case of a pure risky asset as an underlying
    - There is a huge 111% increase for the equity value, but much smaller increases for the VolTarget strategy. This is due to the fact the VolTarget portfolio rebalances into the risk-free asset in high volatility environments
    - Option prices corresponding to higher risky asset volatility levels are significantly lower in the case of the VolTarget portfolio related underlying assets compared to the case of a pure risky asset as an underlying (i.e. 49.60 is much higher than 23.21 and 31.19)
  - For $\sigma = 10\%$, VolTarget ($\bar{V} = 20\%$) has the highest call option price
    - When $\sigma = 10\%$ but $\bar{V} = 20\%$, this leads to a leveraged equity position with weight over 100%
    - The high risky asset weight increases the portfolio volatility and thus the call option value

Asian Call Option Results Linked to VolTarget Portfolios

- We can also look at path-dependent call option payoffs such as arithmetic Asian options
  - Variable annuity writers are exposed to path-dependent risk on VA liabilites
  - Can use volatility target funds to manage risk, but need to understand the path dependent exposures and risks
  - Therefore, supplementing with path-dependent analysis (and not just European) can give helpful insights. Using an arithmetic Asian option is still a simplified approach, but is able to capture path dependency
- In this case, the payoff function is given by (where S is the underlying asset and K is the strike):

$$f(S) = \max\left(\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K, 0\right)$$

- Can repeat the Monte Carlo simulation to value the call options using the arithmetic Asian option payoff equation above
- Using this framework, ATM call option prices can be computed using different risky asset annual volatilies σ:

Underlying Asset	$\sigma = 10\%$	$\sigma = 50\%$	Change
Equity	11.73	27.26	132%
VolTarget ($\bar{V} = 20\%$)	15.05	17.16	14%

- Results
  - Asian options are in general less expensive than plain vanilla options
  - Asian option prices increase with increasing volatility of the underlying risky asset
  - As was in the case with European options, if the underlying asset is related to a VolTarget portfolio, the option prices exhibit lower dependence on the risky asset volatility (i.e. the percentage change is largest for pure equity)
    - Similar to the case of plain call options, one observes lower Asian option prices corresponding to higher risky asset volatility levels when the underlying asset is related to the VolTarget portfolio compared to the case of the pure risky asset as an underlying
    - Options are often expensive in high volatility environments. For the volatility target portfolios, the swings are milder and this may be an attractive feature to lower risk and price fluctuations

Section 4: VolTarget Portfolio Linked Guarantee Structures

Guarantee Structure with Direct Participation in an Underlying Asset (European Options)

- Suppose an investor purchases a guarantee with initial investment K at time $t = 0$ that has some level p_{dp} of participation in upward equity movements
- Then the payoff can be calculated as:

$$P_{dp} = \max\left[K, K\cdot\left(1 + p_{dp}\frac{S_T - S_0}{S_0}\right)\right] = K + K\cdot p_{dp}\cdot\max\left[\frac{S_T}{S_0} - 1, 0\right]$$

- Notation
  - K: original investment amount
  - S_t: value of risky asset (i.e. equity) at time t
  - T: length of time for the guarantee
  - $p_{dp} > 0$: participation rate where the "dp" subscript refers to "direct participation"
  - P_{dp}: payoff at maturity from guarantee structure
- The policyholder is guaranteed to get at least the initial investment K (sometimes called "100% capital protection at maturity")
- Additionally, if equity returns are positive, the payoff increases by the product of the following three terms: initial investment, participation rate and the equity return
- Based on the blue equation above, we can think of this guarantee as a combination of a zero-coupon bond and a call option
  - Can take the K investment and purchase Ke^{-rT} is zero-coupon bonds and invest the remaining $K - Ke^{-rT}$ in a call option

  - $K - Ke^{-rT} = K(1 - e^{-rT})$ is called the "risk budget", which is the amount spent purchasing call options

- Can define g_{dp} as a call option payoff in the following form:

$$g_{dp}(x) = K \max\left[\frac{x}{S_0} - 1, 0\right]$$

- Can also define the option payoff f_{dp} as follows:

$$f_{dp}(x) = \max\left[\frac{x}{S_0} - 1, 0\right]$$

- Obviously, $g_{dp}(x) = Kf_{dp}(x)$
- Next, the reading states an important conclusion. This is not proved but keep in mind the following statement:
  - **The participation rate is equal to the number of units of option g that can be purchased for the given risk budget**
  - In other words, the participation rate equals the risk budget divided by option cost g
  - Also, note that the "blue equation" can be interpreted as the sum of (1) zero-coupon bond value at maturity (i.e. K) plus (2) the payoff at maturity from p_{dp} units of the call option with payoff g_{dp}
- Putting this all together gives:

$$p_{dp} = \frac{\text{Risk Budget}}{V_0(g_{dp})} = \frac{K(1 - e^{-rT})}{V_0(g_{dp})} = \frac{K(1 - e^{-rT})}{KV_0(f_{dp})} = \frac{1 - e^{-rT}}{V_0(f_{dp})}$$

- In recent years, markets have had low interest rates and periods of high volatility:
  - Lower interest rates r result in a higher PV of the guarantee, which leads to a lower risk budget
  - For a fixed interest rate r, a higher volatility of the underlying risky asset leads to a more expensive option price V_0
  - This often means **insurers are unable to offer as competitive of guarantee structures, and lower participation rates are offered when interest rates are low / volatility is high**. This is explored in more detail in the tables below
- The no-arbitrage participation rates can be computed across a range of interest rate and volatility scenarios *assuming the underlying asset is pure equity:*

Interest Rate	$\sigma = 10\%$	$\sigma = 50\%$
$r = 2\%$	$p_{dp} = 67\%$	$p_{dp} = 21\%$
$r = 4\%$	$p_{dp} = 88\%$	$p_{dp} = 37\%$
$r = 6\%$	$p_{dp} = 95\%$	$p_{dp} = 50\%$

- The no-arbitrage participation rates can also be computed across a range of interest rate and volatility scenarios *assuming the underlying asset is a VolTarget portfolio with* $\bar{V} = 20\%$:

Interest Rate	$\sigma = 10\%$	$\sigma = 50\%$
$r = 2\%$	$p_{dp} = 43\%$	$p_{dp} = 40\%$
$r = 4\%$	$p_{dp} = 68\%$	$p_{dp} = 64\%$
$r = 6\%$	$p_{dp} = 80\%$	$p_{dp} = 77\%$

- Key insights from the tables above:
  - Participation rates are highest in a high interest rate, low volatility environment
  - Participation rates are lowest in a low interest rate, high volatility environment. For example, for the pure equity case when $r = 2\%$ and $\sigma = 50\%$, the participation rate offered is very low at only 21%
  - For a risky market environment ($\sigma = 50\%$), VolTarget offers more attractive participation rates
    - Switching from a pure risky asset as an underlying to a VolTarget-related underlying leads to lower option prices in the case of high risky asset volatilities
    - For $\sigma = 50\%$, VolTarget portfolio offers higher participation rates than pure equity

Guarantee Structure with Direct Participation in the Average of the Underlying Asset (Asian Options)

- *This section is very similar to the prior section, but moves from European to Asian options. The key result is that if the guarantee is based on an average return, participation rates increase because purchasing options is cheaper (due to the lower volatility from averaging)*
- First, recall for the European payoff we had that:

$$P_{dp} = \max\left[K, K \cdot \left(1 + p_{dp} \times \frac{S_T - S_0}{S_0}\right)\right]$$

- We can write the Asian/averaging payoff as:

$$P_{dpa} = \max\left[K, K \cdot \left(1 + p_{dpa} \times \frac{\frac{1}{n}\sum_{i=0}^{n} S_{t_i} - S_0}{S_0}\right)\right]$$

- We simply replace S_T with the average $\frac{1}{n}\sum_{i=0}^{n} S_{t_i}$
- At maturity time T, the investor receives at least the initial investment K. Additionally, if equity returns are positive, the payoff increases by the product of the following three terms: initial investment, participation rate and the average equity return

- Here, "dpa" refers to "direct participation with averaging"
- The f and g equations are exactly the same, except that we substitute in $x = S = \frac{1}{n}\sum_{i=0}^{n} S_{t_i}$:
    - $g_{dpa}(S) = K \max\left[\frac{\frac{1}{n}\sum_{i=0}^{n} S_{t_i}}{S_0} - 1, 0\right]$
    - $f_{dpa}(S) = \max\left[\frac{\frac{1}{n}\sum_{i=0}^{n} S_{t_i}}{S_0} - 1, 0\right]$
    - Again, $g_{dpa}(S) = K f_{dpa}(S)$
- The final equations are exactly the same:

$$p_{dp} = \frac{\text{Risk Budget}}{V_0(g_{dpa})} = \frac{K(1-e^{-rT})}{V_0(g_{dpa})} = \frac{K(1-e^{-rT})}{KV_0(f_{dpa})} = \frac{1-e^{-rT}}{V_0(f_{dpa})}$$

- We can again determine the no-arbitrage "fair" participation rates
- No-arbitrage participation rates can be computed across a range of interest rate and volatility scenarios *assuming the underlying asset is pure equity with averaging:*

Interest Rate	$\sigma = 10\%$	$\sigma = 50\%$
$r = 2\%$	$p_{dpa} = 133\%$	$p_{dpa} = 41\%$
$r = 4\%$	$p_{dpa} = 176\%$	$p_{dpa} = 69\%$
$r = 6\%$	$p_{dpa} = 194\%$	$p_{dpa} = 90\%$

- No-arbitrage participation rates can be computed across a range of interest rate and volatility scenarios *assuming the underlying asset is a VolTarget portfolio with* $\bar{V} = 20\%$ *with averaging:*

Interest Rate	$\sigma = 10\%$	$\sigma = 50\%$
$r = 2\%$	$p_{dpa} = 82\%$	$p_{dpa} = 66\%$
$r = 4\%$	$p_{dpa} = 129\%$	$p_{dpa} = 109\%$
$r = 6\%$	$p_{dpa} = 153\%$	$p_{dpa} = 142\%$

- Key insights from the tables above:
    - We see the same result as before that the participation rates are highest for high interest rates and low volatility
    - In fact, in this market environment it is sometimes even theoretically possible to have participation rates above 100%
    - We also see that the participation rates are higher under the averaging approach (compared to European)

  * This makes sense, because Asian options are less volatile than European options due to averaging
  * The lower volatility means that purchasing options is cheaper, and so the participation rate increases
- For $\sigma = 50\%$, VolTarget offers higher participation rates than pure equity. This makes sense, because VolTarget mitigates the volatility to $\bar{V} = 20\%$ and so the options are cheaper and thus participation rates are higher

Guarantee with Equity Participation Example

Consider an initial investment of 100,000 with equity participation of 60%. Suppose equity markets increase by x% over the holding period. What is the payoff for the given values of x below:

(a) $x = 4\%$

(b) $x = -12\%$

Next, suppose you are given continuous interest rate $r = 3\%$ and a time horizon of $T = 1$. If the insurer replicates the guarantee by purchasing a zero-coupon bond and call options, and assuming no-arbitrage, how much should they invest in:

(c) Zero coupon bonds

(d) Options

Solution:

We are given that $K = 100{,}000$ and $p_{dp} = 60\%$. If markets go up, we receive 60% participation in the price appreciation. If equity markets decrease over the holding period, we simply retain our original investment value.

Thus:

(a) Payoff $= 100{,}000 + 100{,}000 \cdot 60\% \cdot 4\% = 102{,}400$

(b) Payoff $= 100{,}000$

(c) $Ke^{-rT} \approx 97{,}044.55$

(d) $RB = K - Ke^{-rT} \approx 2{,}955.45$

Section 5: Conclusion

- Using a VolTarget portfolio as an alternative to a pure equity index investment may improve long-term risk-return portfolio characteristics
- However, there are drawbacks of the VolTarget mechanism:
  1. **Works best in specific market environments**
     - Such as a falling market accompanied by high volatility levels or a rising market accompanied by low volatility levels
  2. **Rule-based nature of the strategy may cause significant losses in nonstandard market environments**
     - For example, large losses could occur in the case of falling and low volatility markets (because of the large equity exposure)
  3. **May not be sufficient to solely define portfolio management decisions and should be combined with other asset allocation strategies**
- Financial options based on the VolTarget portfolio are typically less expensive compared to pure equity as an underlying asset
  - Typically also makes the option price less dependent on the risky asset volatility σ
- VolTarget may be a suitable way for investors to get equity participation with downside protection. This can be attractive for investors simultaneously seeking higher yields and downside protection in market environments with low interest rates and high volatilities
- Main idea behind the VolTarget mechanism is a *controlled risk volatility level*
  - The calculation of equity/bond weights is based on volatility (and not directly based on returns)

