SECTION 2

INTEREST RATES

ACCUMULATION FUNCTION

$a(t)$ = Value of \$1 after t years

$A(t)$ = Value of \$k after t years
= $k[a(t)]$

RATE OF INTEREST

Interest earned in year n:

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)}$$

Compound interest: $a(t) = (1+i)^t$

$a(0) = 1$

$a(1) = 1+i$

Simple interest: $a(t) = (1+it)$

RATE OF DISCOUNT

Discount:
Interest at the beginning of the year

Discount earned in year n:

$$d_n = \frac{A(n) - A(n-1)}{A(n)}$$

RATE OF DISCOUNT

Assuming compound interest, we have

$$d_1 = \frac{(1+i) - 1}{(1+i)}$$

$$= \frac{i}{(1+i)}$$

$$= iv$$

$$d_1 = \frac{(1+i) - 1}{(1+i)}$$

$$= 1 - \frac{1}{(1+i)}$$

$$= 1 - v$$

RATE OF DISCOUNT

Assuming compound interest, we have

$$i - d = i - iv$$
$$= i(1-v)$$
$$= id$$

Compound discount: $a(t) = (1-d)^{-t}$

Simple discount: $a(t) = (1-td)^{-1}$

NOMINAL RATES: INTEREST AND DISCOUNT

Nominal rates - expressed as annual rate, convertible more frequently

$$1+i = \left[1 + \frac{i^{(m)}}{m} \right]^m$$

$$i^{(m)} = \left[(1 + i)^{(1/m)} - 1 \right] m$$

EXAMPLE

Effective rate
12% per annum
$i = 12\%$

Nominal rate
12% per annum, payable monthly
$i^{(12)} = 12\%$
$i = (1.01)^{12} - 1$

NOMINAL RATES: INTEREST AND DISCOUNT

Nominal rates - expressed as annual rate, convertible more frequently

$$(1-d)^{-1} = \left[1 - \frac{d^{(m)}}{m}\right]^{-m} = 1+i$$

$$d^{(m)} = \left[1-(1-d)^{(1/m)}\right]m$$
$$= \left[1-(1+i)^{(-1/m)}\right]m$$

$$\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m}$$

FORCE OF INTEREST AND DISCOUNT

$$i^{(m)} = \left[(1 + i)^{(1/m)} - 1 \right] m$$

$$d^{(m)} = \left[1 - (1 - d)^{(1/m)} \right] m$$

Force of interest is the limiting value of $i^{(m)}$ as the compounding frequency increases:

$$\lim_{m \to \infty} i^{(m)} = \lim_{m \to \infty} d^{(m)} = \delta$$

CALCULUS REVIEW
NATURAL LOGARITHM

ℓn(x) is the natural logarithm of x

Let y = ℓn(x)

dy/dx = 1/x

e is base of the natural logarithm function

Let y = e^x

dy/dx = e^x

CALCULUS REVIEW

Let y = f(x) = g(x)/h(x)

dy/dx = f'(x)

$$= \frac{h(x)*g'(x) - g(x)*h'(x)}{[h(x)]^2}$$

Let y = f(x) = g(h(x))

dy/dx = f'(x)

= g'(h(x))*h'(x)

FORCE OF INTEREST AND DISCOUNT

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a(t) = Value of \$1 after t years

A(t) = Value of \$k after t years
= k[a(t)]

Force of interest is the instantaneous rate of change of the accumulation function

Let y = A(t)

dy/dt = A'(t)

Must modify this to determine force of interest – see next page

FORCE OF INTEREST AND DISCOUNT

Must divide by A(t) to give result independent of amount of deposit:

$$\delta_t = \frac{A'(t)}{A(t)}$$

Same result using a(t) function:

$$\delta_t = \frac{a'(t)}{a(t)}$$

FORCE OF INTEREST AND DISCOUNT

One or two prior exam problems gave A(t), and you had to derive the force of interest:

Let y = *ln*[A(t)]

dy/dt = [1/A(t)]*A'(t)

$$= \frac{A'(t)}{A(t)}$$

$$= \delta_t$$

This is based on page 21:

Let y = f(t) = g(h(t))
dy/dt = f'(t)
= g'(h(t))*h'(t)

FORCE OF INTEREST AND DISCOUNT

Compound interest example:

$a(t) = (1+i)^t$

$a'(t) = (1+i)^t \ln(1+i)$

$\delta_t = \dfrac{a'(t)}{a(t)}$

$\delta_t = \ln(1+i)$ which is a constant

$e^\delta = 1+i$

§430(h)(2)(F)
YIELD CURVE

PRACTICAL NOTE - NOT ON SYLLABUS:

Data is published monthly by IRS via Notices:

- 430(h)(2)(D) yield curve
- 430(h)(2)(C) segment rates
- 417(e)(3)(D)(i) modified yield curve

Some hints on methodology used are in IRS Notice 2007-81

Technical details are in this write-up:
http://www.ustreas.gov/offices/economic-policy/reports/corporate_yield_curve_2007.pdf

YIELD CURVE
SPOT INTEREST RATES

Sample reporting - Yield Curve
NOVEMBER 2008 - IRS Notice 2008-112

Table I

Monthly Yield Curve for November 2008

Maturity	*Yield*	*Maturity*	*Yield*	*Maturity*	*Yield*	*Maturity*	*Yield*	*Maturity*	*Yield*
0.5	4.92	20.5	8.05	40.5	7.35	60.5	7.13	80.5	7.03
1.0	5.93	21.0	8.02	41.0	7.34	61.0	7.13	81.0	7.02
1.5	6.77	21.5	7.98	41.5	7.33	61.5	7.13	81.5	7.02
2.0	7.35	22.0	7.95	42.0	7.33	62.0	7.12	82.0	7.02
2.5	7.65	22.5	7.91	42.5	7.32	62.5	7.12	82.5	7.02
3.0	7.75	23.0	7.88	43.0	7.31	63.0	7.12	83.0	7.02
3.5	7.74	23.5	7.85	43.5	7.30	63.5	7.11	83.5	7.01
4.0	7.70	24.0	7.82	44.0	7.30	64.0	7.11	84.0	7.01
4.5	7.66	24.5	7.79	44.5	7.29	64.5	7.11	84.5	7.01
5.0	7.64	25.0	7.77	45.0	7.28	65.0	7.10	85.0	7.01
5.5	7.64	25.5	7.74	45.5	7.28	65.5	7.10	85.5	7.01
6.0	7.68	26.0	7.72	46.0	7.27	66.0	7.10	86.0	7.00
6.5	7.74	26.5	7.70	46.5	7.26	66.5	7.09	86.5	7.00
7.0	7.81	27.0	7.68	47.0	7.26	67.0	7.09	87.0	7.00
7.5	7.90	27.5	7.66	47.5	7.25	67.5	7.09	87.5	7.00
8.0	7.99	28.0	7.64	48.0	7.25	68.0	7.09	88.0	7.00
8.5	8.08	28.5	7.62	48.5	7.24	68.5	7.08	88.5	7.00
9.0	8.17	29.0	7.61	49.0	7.24	69.0	7.08	89.0	6.99
9.5	8.25	29.5	7.59	49.5	7.23	69.5	7.08	89.5	6.99
10.0	8.31	30.0	7.58	50.0	7.23	70.0	7.07	90.0	6.99
10.5	8.37	30.5	7.56	50.5	7.22	70.5	7.07	90.5	6.99

Individual rates are spot rates – yield for zero coupon bond of same maturity

PRESENT VALUES USING YIELD CURVE

In general,

$$PV = \sum_{t=0}^{\omega} (1+i)^{-t} {}_tp_x^{(T)} (\text{Benefit Payment}_{x+t})$$

Yield curve – interest rates vary each year:

$$\sum_{t=0}^{\omega} (1+i_t)^{-t} {}_tp_x^{(T)} (\text{Benefit Payment}_{x+t})$$

Note subscript on i in second summation

PRESENT VALUES USING YIELD CURVE FORWARD INTEREST RATES

Derive forward rates k_t equivalent to the yield curve rates i_t

$(1+i_t)^{-t} = [(1+k_1)(1+k_2)(1+k_3) \ \dots \ (1+k_t)]^{-1}$

$(1+i_1)^{-1} = [(1+k_1)]^{-1}$
$(1+i_2)^{-2} = [(1+k_1)(1+k_2)]^{-1}$
$(1+i_3)^{-3} = [(1+k_1)(1+k_2)(1+k_3)]^{-1}$

PRESENT VALUES USING YIELD CURVE FORWARD INTEREST RATES

Yield curve – interest rates vary each year:

$$\sum_{t=0}^{\omega} (1+i_t)^{-t} \, {}_t p_x^{(T)} (\text{Benefit Payment}_{x+t})$$

Forward rates:

$$\sum_{t=0}^{\omega} [(1+k_1)(1+k_2)\ldots(1+k_t)]^{-1} \, {}_t p_x^{(T)} (\text{Benefit Payment}_{x+t})$$

Can use forward rates in identical manner as select and ultimate rates