Hurlimann

Tasks:

ii.1 Calculate unpaid claims estimates

ii.2 Test unpaid claim estimates for reasonableness

iii.1 Estimate parameters of unpaid claims distributions

iv.1 Test output of unpaid claim distributions for reasonableness

Hurlimann's credible loss ratio method for estimating reserves is a credibility weighting of the chain ladder and Bornhuetter-Ferguson reserves. Sound familiar? Yes, this is similar to the Benktander method. In fact, we're going to see the Benktander method reappear. The difference is in how we calculate the credibility factor, including the calculation of the percent paid or reported to date.

Advantages:

1. Calculations are straightforward.
2. As long as the same premiums are used, different actuaries will always achieve the same results for the collective reserve – this is also true for the individual reserve, without restriction.
3. Using the optimal credibility weights minimizes both the MSE and the variance of the claims reserve.

Notation

For origin period i reported in period k:

n = number of origin periods in the triangle

S_{ik} = incremental paid claims

C_{ik} = cumulative paid claims

R_i = reserves

U_i = ultimate claims

V_i = premium

m_k = incremental loss ratio

LR = total loss ratio ($= \sum_k m_k$)

p_i = loss ratio payout factor (or loss ratio lag factor or percent paid)

q_i = loss ratio reserve factor (percent unpaid)

U_i^{BC} = burning cost estimate of ultimate claims ($= V_i \times LR$)

Assumptions

The following assumptions are necessary in using Hurlimann's method:

- Losses are fully developed at the end of the triangle (i.e. the tail factor is 1.0).
- $Var(U_i) = Var(U_i^{BC})$
  - This is Hurlimann's notation for saying that the variance of the ultimate using his method should equal the variance of the ultimate using the expected loss method (U_i^{BC} = ELR x premium).
  - When you see this assumption in an exam question, it should be a clue that you need to use Hurlimann's method.
- $E(U_i) = E(U_i^{BC})$, which is just a fancy way of saying that the ultimate should be unbiased.
- Given the true ultimate U_i, the percent paid to date is assumed to have an expected value of p_i and a variance of $p_i q_i \beta_i^2(U_i)$.
  - $\beta_i^2(U_i)$ is a shape parameter describing how much the percent paid to date varies around its expected value p_i, for a given true ultimate U_i.
  - This is where $\alpha_i^2(U_i)$, used throughout the MSE formulas and in the Estimating $E[\alpha_i^2(U_i)]$ section, comes from: $\alpha_i^2(U_i) = U_i^2 \beta_i^2(U_i)$.

Calculating Hurlimann Ultimate Losses

Example: You are given the following incremental loss development triangle and earned premiums for a book of business. Calculate the Benktander, Neuhaus, and Optimal Credibility reserves for each year. Assume $Var(U_i) = Var(U_i^{BC})$.

	1	2	3	4	Premium
2020	5,525	2,500	875	475	10,250
2021	4,500	2,600	900		9,900
2022	4,750	2,560			10,000
2023	4,900				9,500

Step 1: Calculate m, the incremental loss ratio.

$$m_k = \frac{E\left[\sum_{i=1}^{n-k+1} S_{ik}\right]}{\sum_{i=1}^{n-k+1} V_i}, \qquad k = 1, ..., n$$

$$\text{incremental loss ratio} = \frac{\text{sum of incremental losses in a reporting period}}{\text{sum of the corresponding premiums}}$$

$$m_1 = \frac{5{,}525 + 4{,}500 + 4{,}750 + 4{,}900}{10{,}250 + 9{,}900 + 10{,}000 + 9{,}500} = 0.49622$$

$$m_2 = \frac{2{,}500 + 2{,}600 + 2{,}560}{10{,}250 + 9{,}900 + 10{,}000} = 0.25406$$

$$m_3 = \frac{875 + 900}{10{,}250 + 9{,}900} = 0.08809$$

$$m_4 = \frac{475}{10{,}250} = 0.04634$$

We can also calculate the total loss ratio, which we will use in future steps, as

$$LR = \sum_{i=1}^{n} m_k = 0.49622 + 0.25406 + 0.08809 + 0.04634 = 0.88471$$

Step 2: Calculate p and q, the percent reported and the percent unreported.

$$p_i = \frac{\sum_{k=1}^{n-i+1} m_k}{\sum_{k=1}^{n} m_k}, \qquad i = 1, ..., n$$

$$\text{percent reported} = \frac{\text{sum of incremental loss ratios for reported periods}}{\text{total loss ratio}}$$

$$p_{2020} = \frac{0.49622 + 0.25406 + 0.08809 + 0.04634}{0.88471} = 1.0$$

$$p_{2021} = \frac{0.49622 + 0.25406 + 0.08809}{0.88471} = 0.94762$$

$$p_{2022} = \frac{0.49622 + 0.25406}{0.88471} = 0.84805$$

$$p_{2023} = \frac{0.49622}{0.88471} = 0.56088$$

$$q_i = 1 - p_i = \text{percent unreported}$$

$$q_{2020} = 0.0$$

$$q_{2021} = 0.05238$$

$$q_{2022} = 0.15195$$

$$q_{2023} = 0.43912$$

Step 3: Calculate the chain ladder reserves, also known as the individual reserves.

Note that Hurlimann calls these the individual reserves because they depend solely on loss experience from an individual *origin period.*

$$R^{ind} = \frac{q_i}{p_i} C_{i,n-i+1}, \qquad i = 1, ..., n$$

$$\text{Reserves} = \frac{\text{percent unreported}}{\text{percent reported}} \text{ x cumulative losses reported}$$

AY	p	q	C	R^{ind}
2020	1.00000	0.00000	9,375	$\frac{0.00000}{1.00000} \times 9{,}375 = 0$
2021	0.94762	0.05238	8,000	$\frac{0.05238}{0.94762} \times 8{,}000 = 442$
2022	0.84805	0.15195	7,310	$\frac{0.15195}{0.84805} \times 7{,}310 = 1{,}310$
2023	0.56088	0.43912	4,900	$\frac{0.43912}{0.56088} \times 4{,}900 = 3{,}836$

Step 4: Calculate the Bornhuetter-Ferguson reserves, also known as the collective reserves.

Note that Hurlimann calls these the collective reserves because they depend on the loss experience of the entire portfolio over all origin periods (i.e. the loss ratio is calculated using experience from all periods).

$$R^{coll} = q_i V_i \sum_{k=1}^{n} m_k, \qquad i = 1, ..., n$$

$$\text{Reserves} = \text{percent unreported x premium x total loss ratio}$$

AY	q	V	R^{coll}
2020	0.00000	10,250	$0.00000 \times 10{,}250 \times 0.88471 = 0$
2021	0.05238	9,900	$0.05238 \times 9{,}900 \times 0.88471 = 459$
2022	0.15195	10,000	$0.15195 \times 10{,}000 \times 0.88471 = 1{,}344$
2023	0.43912	9,500	$0.43912 \times 9{,}500 \times 0.88471 = 3{,}691$

Step 5: Find Z, the credibility factor. We have 3 different options:

a Gunnar Benktander

$$Z_i^{GB} = p_i$$

b Walter Neuhaus

$$Z_i^{WN} = p_i \sum_{k=1}^{n} m_k, \qquad i = 1, ..., n$$

c Optimal Credibility

$$Z_i^{Opt} = \frac{p_i}{p_i + t_i^*} \qquad \text{where } t_i^* = \frac{f_i - 1 + \sqrt{(f_i+1)(f_i-1+2p_i)}}{2}, \qquad i = 1, ..., n$$

f_i represents how many times more volatile the true ultimate U_i is than the burning cost (expected loss ratio based) estimate: $Var(U_i) = f_i \times Var(U_i^{BC})$, with $f_i \geq 1$ since the true ultimate can only be at least as volatile as the burning cost estimate, never less. A larger f_i produces a larger t_i^* and therefore a smaller Z_i^{Opt} – more volatility in the true ultimate beyond what the burning cost estimate already reflects means we trust the individual reserve less, shifting more weight to the collective reserve.

Under the assumption that $Var(U_i) = Var(U_i^{BC})$, $f_i = 1 \Rightarrow t_i^* = \sqrt{p_i}$, and $Z_i^{Opt} = \frac{p_i}{p_i + \sqrt{p_i}}$

AY	p	Z^{GB}	Z^{WN}	Z^{Opt}
2020	1.00000	1.00000	$1.00000 \times 0.88471 = 0.88471$	$\frac{1.00000}{1.00000 + \sqrt{1.00000}} = 0.50000$
2021	0.94762	0.94762	$0.94762 \times 0.88471 = 0.83837$	$\frac{0.94762}{0.94762 + \sqrt{0.94762}} = 0.49328$
2022	0.84805	0.84805	$0.84805 \times 0.88471 = 0.75028$	$\frac{0.84805}{0.84805 + \sqrt{0.84805}} = 0.47941$
2023	0.56088	0.56088	$0.56088 \times 0.88471 = 0.49622$	$\frac{0.56088}{0.56088 + \sqrt{0.56088}} = 0.42822$

Notice that when $f_i = 1$, $Z^{Opt} \leq 0.5$. Z_i will typically be higher for Benktander and Neuhaus than for the optimal credibility method.

All three credibility choices are really the same formula, $Z_i = \frac{p_i}{p_i + t_i}$, just with a different t_i plugged in:

$$t_i^{GB} = q_i$$

$$t_i^{WN} = q_i + \frac{1 - LR}{LR}$$

$$t_i^* = \frac{f_i - 1 + \sqrt{(f_i+1)(f_i-1+2p_i)}}{2} \qquad \text{(the Optimal Credibility } t_i^* \text{ from above)}$$

Step 6: Calculate the credibility weighted reserves.

$$R^c = Z \times R^{ind} + (1 - Z) \times R^{coll}$$

Credible Loss Ratio Reserves = Z x Chain Ladder Reserves + (1 - Z) x Bornhuetter-Ferguson Reserves

AY	R^{ind}	R^{coll}	R^{GB}	R^{WN}	R^{Opt}
2020	0	0	0	0	0
2021	442	459	443	445	451
2022	1,310	1,344	1,315	1,318	1,328
2023	3,836	3,691	3,772	3,763	3,753

Further Iterations

Just like the Benktander method, we can continue to iterate the credible loss ratio reserve using the rule

$$R_i^{(m)} = q_i \times U_i^{(m)}, \qquad U_i^{(m+1)} = C_{i,n-i+1} + R_i^{(m)}$$

Starting this iteration at the expected ultimate under the loss ratio method alone gives:

$$U_i^{(0)} = V_i \times LR$$

$$U_i^{(1)} = U_i^{coll}$$

$$\dots$$

$$U_i^{(m)} = (1 - q_i^m) U_i^{ind} + q_i^m U_i^{(0)}$$

$$\dots$$

$$U_i^{(\infty)} = U_i^{ind}$$

Mean Squared Error

An estimate of reserves isn't very useful unless we have some confidence that it will be close to the actual future losses. The mean squared error allows us to have some confidence in our estimates. The mean squared error = $(\text{actual} - \text{expected})^2$. A small MSE means that our estimate is close to the actual future losses. The MSEs of the reserves can be calculated generally as follows:

$$mse(R_i) = E[\alpha_i^2(U_i)] \left(\frac{Z_i^2}{p_i} + \frac{1}{q_i} + \frac{(1 - Z_i)^2}{t_i} \right) q_i^2$$

If we want to calculate the MSE of the individual reserves, we set $Z = 1$, which allows us to simplify the MSE formula to the following:

$$mse(R_i^{ind}) = E[\alpha_i^2(U_i)] \frac{q_i}{p_i}$$

If we want to calculate the MSE of the collective reserves, we set $Z = 0$, which allows us to simplify the MSE formula as follows:

$$mse(R_i^{coll}) = E[\alpha_i^2(U_i)]q_i\left(1 + \frac{q_i}{t_i}\right)$$

$E[\alpha_i^2(U_i)]$ might be given to you directly on the exam, but it can also be calculated from the same triangle used to calculate the reserves, as long as we're willing to assume a value for f_i (the same f_i used to calculate the optimal credibility weight).

Example: Using the same information given in the prior example, calculate $E[\alpha_{2023}^2(U_{2023})]$, assuming $f_i = 1$.

Step 1: Calculate s_k^2, the variance of the individual origin periods' incremental loss ratios around m_k, for each development period.

$$s_k^2 = \frac{1}{n-k}\sum_{i=1}^{n-k+1} V_i\left(\frac{S_{ik}}{V_i} - m_k\right)^2, \qquad k = 1, ..., n-1$$

$$\text{variance of incr. LRs} = \frac{1}{df}\sum \text{premium} \times \left(\frac{\text{incr. losses}}{\text{premium}} - \text{incr. LR}\right)^2$$

There isn't enough data to calculate s_n^2 directly (the last development period only has one origin period's worth of data), so we set it equal to the smallest of the other s_k^2 values:

$$s_n^2 = \min\{s_k^2 : k = 1, ..., n-1\}$$

k	w_k	m_k	s_k^2
1	39,650	0.49622	14.70509
2	30,150	0.25406	0.91083
3	20,150	0.08809	0.15474
4	10,250	0.04634	0.15474 (tail, $= s_3^2$)

where $w_k = \sum_{i=1}^{n-k+1} V_i$ = the same denominator already used to calculate m_k in Step 1.

Step 2: Calculate the burning cost estimate and its variance.

$$U_i^{BC} = V_i \times LR$$

$$U_{2023}^{BC} = 9{,}500 \times 0.88471 = 8{,}404.75$$

$$Var(U_i^{BC}) = V_i^2\left[\sum_{k=1}^{n}\frac{s_k^2}{w_k}\right]$$

$$\sum_{k=1}^{n} \frac{s_k^2}{w_k} = \frac{14.70509}{39{,}650} + \frac{0.91083}{30{,}150} + \frac{0.15474}{20{,}150} + \frac{0.15474}{10{,}250} = 0.00042386$$

$$Var(U_{2023}^{BC}) = 9{,}500^2 \times 0.00042386 = 38{,}253$$

Step 3: Calculate $\hat{u}$, an estimate of the inverse of the coefficient of variation squared of U_i^{BC} – this is the same for every origin period.

$$\hat{u} = \frac{(U_i^{BC})^2}{Var(U_i^{BC})} = \frac{LR^2}{\sum_{k=1}^{n} s_k^2/w_k}$$

$$\hat{u} = \frac{0.88471^2}{0.00042386} = 1{,}846.64$$

Step 4: Back out β_i^2 using the same f_i and t_i^* used for the optimal credibility weight (recall $t_i^* = \sqrt{p_i}$ under our assumption that $f_i = 1$).

$$t_{2023}^* = \sqrt{p_{2023}} = \sqrt{0.56088} = 0.74892$$

$$\beta_i^2 = \frac{1 + f_i}{f_i + \hat{u}} \times \frac{t_i^*}{1 + t_i^*}$$

$$\beta_{2023}^2 = \frac{1+1}{1+1{,}846.64} \times \frac{0.74892}{1.74892} = 0.00046353$$

Step 5: Calculate $E[\alpha_i^2(U_i)]$.

$$E[\alpha_i^2(U_i)] = \beta_i^2 \left(f_i \times Var(U_i^{BC}) + (U_i^{BC})^2\right)$$

$$E[\alpha_{2023}^2(U_{2023})] = 0.00046353 \times \left(1 \times 38{,}253 + 8{,}404.75^2\right) = 32{,}761$$

Now that we know how to calculate $E[\alpha_i^2(U_i)]$, let's look at how we can use that to calculate MSE.

Example: Using the information from the prior examples and $t_i = \sqrt{p_i}$, calculate the MSE for the individual, collective, and optimal credibility reserves for 2023.

From the prior examples, we have the following:

$p_{2023} = 0.56088$

$q_{2023} = 0.43912$

$Z^{Opt} = 0.42822$

From that, we can calculate

$$mse(R_i^{ind}) = 32{,}761 \times \frac{0.43912}{0.56088} = 25{,}649$$

$$mse(R_i^{coll}) = 32{,}761 \times 0.43912 \times \left(1 + \frac{0.43912}{\sqrt{0.56088}}\right) = 22{,}821$$

$$mse(R_i^{opt}) = 32{,}761 \times \left(\frac{0.42822^2}{0.56088} + \frac{1}{0.43912} + \frac{(1-0.42822)^2}{\sqrt{0.56088}}\right) \times 0.43912^2 = 19{,}209$$

Notice that the optimal credibility method has the lowest MSE. It always will. The value of f_i in the optimal credibility weights is calculated in such a way that the MSE of those reserves will always be minimized.

Hurlimann tells us the MSEs for the reserves based on the Benktander and Neuhaus credibility weights will usually be lower than either the individual or collective reserve MSEs, but higher than the optimal credibility reserve MSE. Interestingly enough, though, they aren't much higher than the MSE for the optimal credibility method. This tells us that if it's not possible to calculate the optimal credibility weights, either the Benktander or Neuhaus methods are good alternatives.

Alternate Z Calculation

The Optimal Credibility weight Z_i^* can also be derived directly from the covariance and variance of the underlying random variables, before we ever introduce f_i, β_i^2, or t_i^*. Here, R_i refers to the true (unknown) reserve, $U_i - C_{i,n-i+1}$ – not any particular model's reserve estimate.

$$Z_i^* = \frac{p_i}{q_i} \times \frac{Cov(C_{i,n-i+1}, R_i) + p_i q_i \times Var(U_i^{BC})}{Var(C_{i,n-i+1}) + p_i^2 \times Var(U_i^{BC})}$$

Using the same conditional model underlying the rest of this method, the covariance and variance terms can be written as:

$$Var(C_{i,n-i+1}) = p_i \times E[\alpha_i^2(U_i)] + p_i^2 \times \left(Var(U_i) - E[\alpha_i^2(U_i)]\right)$$

$$Cov(C_{i,n-i+1}, R_i) = p_i q_i \times \left(Var(U_i) - E[\alpha_i^2(U_i)]\right)$$

where $Var(U_i) = f_i \times Var(U_i^{BC})$.

Example: Using the information from the prior examples, calculate Z_{2023}^* using the variance/covariance method, and confirm that it matches the value calculated previously.

Step 1: Calculate $Var(U_{2023})$, $Var(C_{2023,4})$, and $Cov(C_{2023,4}, R_{2023})$, the building blocks of the Z_i^* formula.

These are built from quantities we already calculated: $E[\alpha_{2023}^2(U_{2023})] = 32{,}761$ *and* $Var(U_{2023}^{BC}) = 38{,}253$.

$$Var(U_{2023}) = 1 \times 38{,}253 = 38{,}253$$

$$Var(C_{2023,4}) = 0.56088 \times 32{,}761 + 0.56088^2 \times (38{,}253 - 32{,}761) = 20{,}103$$

$$Cov(C_{2023,4}, R_{2023}) = 0.56088 \times 0.43912 \times (38{,}253 - 32{,}761) = 1{,}353$$

Step 2: Calculate Z^*_{2023}.

$$Z^*_{2023} = \frac{0.56088}{0.43912} \times \frac{1{,}353 + 0.56088 \times 0.43912 \times 38{,}253}{20{,}103 + 0.56088^2 \times 38{,}253} = 0.42822$$

This matches $Z^{Opt}_{2023} = 0.42822$, calculated earlier using t^*_i – the two approaches are mathematically equivalent, just built from different pieces.

Reinterpreting the Standard Methods

Hurlimann's framework isn't limited to loss-ratio-based factors – if we instead use the percent paid and percent unpaid from a standard chain-ladder analysis, p^{CL}_i and q^{CL}_i (calculated the usual way, from selected age-to-age factors), the Chain-Ladder, Cape Cod, and Bornhuetter-Ferguson methods can all be written as special cases of the credible loss ratio reserve.

The Chain-Ladder Method is exactly the individual reserve, using chain-ladder factors in place of loss ratio factors:

$$R^{CL}_i = \frac{q^{CL}_i}{p^{CL}_i} \times C_{i,n-i+1}$$

The Cape Cod Method is a Benktander-type credibility mixture, with

$$R^{ind}_i = \frac{q^{CL}_i}{p^{CL}_i} \times C_{i,n-i+1}, \quad R^{coll}_i = q^{CL}_i \times LR \times V_i, \quad LR = \frac{\sum_{i=1}^{n} C_{i,n-i+1}}{\sum_{i=1}^{n} p^{CL}_i \times V_i}, \quad Z_i = p^{CL}_i$$

Notice that LR here is a single loss ratio calculated from the data (total losses to date over total "used-up" premium) – the same idea as the Cape Cod method you already know.

The Optimal Cape Cod Method uses the same mixture, but with the optimal credibility weight instead of $Z_i = p^{CL}_i$:

$$Z_i = \frac{p^{CL}_i}{p^{CL}_i + \sqrt{p^{CL}_i}}$$

The Bornhuetter-Ferguson Method uses the same credibility mixture as Cape Cod, except LR_i is a separately *selected* expected loss ratio for each origin period, rather than one blended value calculated from the whole triangle:

$$R_i^{ind} = \frac{q_i^{CL}}{p_i^{CL}} \times C_{i,n-i+1}, \qquad R_i^{coll} = q_i^{CL} \times LR_i \times V_i, \qquad Z_i = p_i^{CL}$$

The Optimal Bornhuetter-Ferguson Method again swaps in the optimal credibility weight from the Optimal Cape Cod method above.

This means Chain-Ladder, Cape Cod, and BF all sit on the same spectrum as Hurlimann's own method – they differ only in which percent-paid factor is used (p_i^{CL} vs. the loss-ratio-based p_i) and how the credibility weight Z_i and collective estimate are chosen.

Problem Knowledge Checklist

1. Be able to recognize when to use Hurlimann's method
2. Be able to calculate reserves using Hurlimann's method
   - Calculate incremental and total loss ratios
   - Calculate the percent paid/reported
   - Calculate the individual (chain ladder) and collective (BF) reserves
   - Know the three different options for the credibility factor and be able to calculate Z
     - Benktander
     - Neuhaus
     - Optimal Credibility
   - Recognize that Benktander, Neuhaus, and Optimal Credibility are all the same form, $Z_i = p_i/(p_i + t_i)$, with a different t_i
   - Be able to explain what the parameter f_i represents in the optimal credibility formula, and how it affects the optimal credibility weight Z_i^{Opt}
   - Be able to explain what $\beta_i^2(U_i)$ represents in the conditional model underlying Hurlimann's method
   - Be able to calculate the Optimal Credibility weight using the variance/covariance method, and recognize it's equivalent to the t_i^* shortcut
   - Calculate credibility weighted reserves
3. Be able to calculate the mean squared error (MSE) for the individual, collective, and optimal credibility reserves
4. Be able to calculate $E[\alpha_i^2(U_i)]$ from a full triangle, given an assumed value of f_i
5. Be able to interpret the MSE
6. Be able to state the advantages of Hurlimann's method
7. Be able to explain the iterative relationship between the collective and individual loss ratio methods, and why continued iteration converges to the individual method
8. Be able to recognize how the Chain-Ladder, Cape Cod, and Bornhuetter-Ferguson methods relate to Hurlimann's credible loss ratio framework